Weak solution to a class of quasilinear elliptic System in Orlicz-Sobolev Spaces

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***Abstract*—The purpose of this paper is to investigate on the existence of a weak solution to the following quasilinear system driven by the M-Laplacian**

$$\begin{cases}(-\Delta_{m_1})u = F_u(x,u,v) & in \quad \Omega,\\ (-\Delta_{m_2})v = F_v(x,u,v) & in \quad \Omega,\\ u=v=0 & on \quad \partial\Omega,\end{cases} \tag{.1}$$

where Ω is a bounded open subset in $\mathbb{R}^N$ and $(-\Delta_m)$ is the M-Laplacian operator.

***Index Terms*—Orlicz-Sobolev spaces , M-Laplacian Operator, Variational problem, Elliptic system.**

I. Introduction

A natural question is to see what results can be recovered when the standard Laplace operator is replaced by the fractional m-Laplacian. In the recent years many others has been an increasing interest in studying non-local problems with p-structure due to its accurate description of models involving anomalous diffusion.

This type of operators arises in many different applications, such as, continuum mechanics, phase transition phenomena, population dynamics, minimal surfaces and game theory, as they are the typical outcome of stochastically stabilization of Levy processes, see for example [7], [12].

In this paper we deal with the existence of a solution to the following quasilinear elliptic system problem

$$\begin{cases}(-\Delta_{m_1})u = F_u(x,u,v) & in \quad \Omega,\\ (-\Delta_{m_2})v = F_v(x,u,v) & in \quad \Omega,\\ u=v=0 & on \quad \partial\Omega,\end{cases} \tag{I.1}$$

where Ω is a bounded open subset in $\mathbb{R}^N$ and $(-\Delta_{m_i})$ is the M-Laplacian operator defined by

$$(-\Delta_{m_i})u := -\text{div}(m_i(|\nabla u|\nabla u)), \quad i=1,2. \tag{I.2}$$

When we take $m_1(t)=|t|^{p-2}$, $m_2(t)=|t|^{q-2}$ $(p,q>1)$. Then the system (I.1) reduces to the following (p,q)-Laplacian system :

$$\begin{cases}(-\Delta)_p u = F_u(x,u,v) & in\,\Omega,\\ (-\Delta)_q v = F_v(x,u,v) & in\,\Omega,\\ u=v=0 & on\,\partial\Omega.\end{cases} \tag{I.3}$$

The existence of solutions for systems like (I.3) have also received a wide range of interests. For this we find in the literature have many researchers have studied this type of systems using some important methods, such as variational method, Nehari manifold and fibering method, three critical points theorem (see for instance [2]–[4]).

In [13], Huentutripay-Manásevich studied an eigenvalue problem to the following system:

$$\begin{cases}-div(m_1(|\nabla u|)\nabla u) = \lambda F_u(x,u,v) & in\,\Omega,\\ -div(m_2(|\nabla v|)\nabla v) = \lambda F_v(x,u,v) & in\,\Omega,\\ u=v=0 & on\,\partial\Omega.\end{cases}$$

For a certain λ, the authors translated the existence of solution into a suitable minimizing problem and proved the existence of solution under some reasonable restriction.

Liben, Zhang and Fang in [15] studied the problem (I.1) by using the Mountain Pass Theorem and they obtained the following result:

Theorem 1.1: [Theorem 3.1 [15]] Assumes that the following conditions hold:

$(\phi_1)'$: $m_i \in C(0,+\infty)$; $tm_i(t) \to 0 \; as\, t \to 0$; $tm_i(t) \to +\infty \; as\, t \to +\infty$.

$(\phi_2)'$: $tm_i(t)$ are strictly increasing.

$(\phi_3)'$:

$$1 < l_i := \inf_{t>0} \frac{t^2 m_i(t)}{M(t)} \leq \sup_{t>0} \frac{t^2 m_i(t)}{M(t)} := n_i < N,$$

where

$$M_i(t) = \int_0^{|t|} s m_i(s) ds, \quad for\ all\ t \in \mathbb{R},$$

and F satisfies:

$(F_0)'$: $F : \Omega \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is a C^1 function such that $F(x,0,0)=0$, for all $x \in \Omega$.

$(F_1)'$: There exist two continuous functions $\Psi_{i=1,2}$: $[0,+\infty) \to \mathbb{R}$, which satisfy that

$$\Psi_i(t) := \int_0^{|t|} \psi_i(s) ds, \quad for\ all\ t \in \mathbb{R},$$

are two N-functions increasing essentially more slowly than $M^*_{i=1,2}$ near infinity, respectively, where M^*_i is the Sobolev

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conjugate function of M_i, which will be specified later. Moreover,

$$n_i < l_{\Psi_i} := \inf_{t>0} \frac{t\psi_i(t)}{\Psi_i(t)} \leq \sup_{t>0} \frac{t\psi_i(t)}{\Psi_i(t)} := n_{\Psi_i} < \infty,$$

such that

$$\begin{cases} |F_u((x,u,v)| \leq c_1(1+\psi_1(|u|)+\overline{\Psi}_1^{-1}(\Psi_2(v))), \\ |F_v((x,u,v)| \leq c_1(1+\psi_2(|v|)+\overline{\Psi}_2^{-1}(\Psi_1(u))), \end{cases}$$

for all $(x,u,v) \in \Omega \times \mathbb{R} \times \mathbb{R}$, where c_1 is a positive constant , $\overline{\Psi}$ denote the complements of $\Psi_{i=1,2}$, respectively.
$(F_2)'$

$$\lim_{|(u,v)|\to+\infty} \frac{F(x,u,v)}{M_1(u)+M_2(v)} = \infty, \qquad \text{uniformly for all x in } \Omega.$$

$(F_3)'$:

$$\lim_{|(u,v)|\to 0} \sup \frac{|F(x,u,v)|}{\lambda_1 m_1(u)+\lambda_2 m_2(v)} = c_0$$

$(F_4)'$: There exists a continuous function $\gamma : [0,\infty) \to \mathbb{R}$ such that

$$\Gamma(t) := \int_0^{|t|} \gamma(s)ds, \quad for\ all\ \ t \in \mathbb{R},$$

is an N-function with

$$1 < l_\Gamma := \inf_{t>0} \frac{t\gamma(t)}{\Gamma(t)} \leq \sup_{t>0} \frac{t\gamma(t)}{\Gamma(t)} := n_\Gamma < +\infty,$$

and functions $H_i(t) := |t|^{\frac{l_i l_\Gamma}{l_\Gamma - 1}}$, $t \in \mathbb{R}$ increase essentially more slowly than M_i^* near infinity, respectively, such that

$$\Gamma\left(\frac{F(x,u,v)}{|u|^{l_1}+|v|^{l_2}}\right) \leq c_2 \overline{F}(x,u,v), \quad x \in \mathbb{R}, \quad |(u,v)| \geq r,$$

where c_2, r are tow strictly positive constants and

$$\overline{F}(x,u,v) := \frac{1}{n_1} F_u(x,u,v)u + \frac{1}{n_2} F_v(x,u,v)v - F(x,u,v),$$

$\forall (x,u,v) \in \Omega \times \mathbb{R} \times \mathbb{R}$. Then the flowing system has a nontrivial solution

$$\begin{cases} -div(m_1(|\nabla u|)\nabla u) = F_u(x,u,v) & in\ \Omega, \\ -div(m_2(|\nabla v|)\nabla v) = F_v(x,u,v) & in\ \Omega, \\ u = v = 0 & on\ \partial\Omega. \end{cases}$$

We get motivated by Theorem 1.1 above and by relaxing hypotheses $(\phi_3)'$, $(F_1)'$ and $(F_2)'$ we shall prove the existence of solution to our problem (I.1).

Let $M\colon \mathbb{R} \to \mathbb{R}^+$ be an N-function, i. e., M is an even and convex function such that

$$M(t) > 0 \quad \text{for } t \neq 0, \quad \lim_{t\to 0} \frac{M(t)}{t} = 0 \quad \text{and} \quad \lim_{t\to\infty} \frac{M(t)}{t} = \infty.$$

Equivalently, M admits the representation:

$$M(t) = \int_0^{|t|} m(s)\,\mathrm{d}s,$$

where $m\colon \mathbb{R}^+ \to \mathbb{R}^+$ is a non-decreasing and right continuous function, with

$$m(0) = 0, \quad m(t) > 0 \quad \text{for } t > 0 \text{ and } \lim_{t\to\infty} m(t) = \infty. \tag{I.4}$$

The N-function $\overline{M}$ complementary to M is defined by $\overline{M}(t) = \int_0^{|t|} \overline{m}(s)\,\mathrm{d}s$, where $\overline{m}\colon \mathbb{R}^+ \to \mathbb{R}^+$ satisfies (I.4). The relationship that relates M and $\overline{M}$ is shown in

$$\overline{M}(t) := \sup_{t\geq 0}\{ts - M(t)\}. \tag{I.5}$$

We shall show that the representation giving by

$$M_{i=1;2}(t) := \int_0^{|t|} rm_i(r)dr \ \ for\ all\ t \in \mathbb{R}, \tag{I.6}$$

where $m_{i=1;2}$ verified (I.4) exists and it's an N-functions .

Proof 1.2: By theorem 1.1 in [11]. Every convex function H which satisfies the condition $H(a) = 0$ can be represented in the form $H(t) = \int_a^{|t|} h(r)dr \quad for\ all\ t \in \mathbb{R}$, where $h(t)$ is a non-decreasing right-continuous function. Note that $h(r) = rm(r)$ then we have by definition of m in (I.4) that h is a non-decreasing and right continuous function for all $t \geq 0$ and we have that

$$h(0) = 0, \quad h(t) = tm(t) > 0 \quad \text{for } t > 0 \text{ and } \lim_{t\to\infty} h(t) = \lim_{t\to\infty} tm(t) \tag{I.7}$$

Then M_i defined in (I.6) is an N-function.
We suppose through our paper that M_i above are satisfying Δ_2-condition globally. Then by lemma 2.5 we have for all $t > 0$ that

$$1 < l_i := \inf_{t>0} \frac{t^2 m_i(t)}{M_i(t)} \leq \sup_{t>0} \frac{t^2 m_i(t)}{M_i(t)} := n_i < N \tag{I.8}$$

Related to function F our hypotheses are the following:
F satisfies:
(F_1): $F : \Omega \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is a C^1 function such that $F(x,0,0) = 0$ for all $x \in \Omega$ and there exist two N-functions increasing essentially more slowly than $M_{i=1,2}$ near infinity $\Psi_{i=1,2} : \mathbb{R} \to \mathbb{R}^+$, which satisfy that

$$n_i < l_{\Psi_i} := \inf_{t>0} \frac{t\psi_i(t)}{\Psi_i(t)} \leq \sup_{t>0} \frac{t\psi_i(t)}{\Psi_i(t)} := n_{\Psi_i} < \infty, \tag{I.9}$$

where

$$\Psi_i(t) := \int_0^{|t|} \psi_i(r)dr, \quad for\ all\ t \in \mathbb{R}.$$

Moreover,

$$\begin{cases} |F_u((x,u,v)| \leq c_1(1+\psi_1(|u|)+\overline{\Psi}_1^{-1}(\Psi_2(v))), \\ |F_v((x,u,v)| \leq c_1(1+\psi_2(|v|)+\overline{\Psi}_2^{-1}(\Psi_1(u))), \end{cases} \tag{I.10}$$

for all $(x,u,v) \in \Omega \times \mathbb{R} \times \mathbb{R}$, where c_1 is a positive constant , $\overline{\Psi}$ denote the complements of $\Psi_{i=1,2}$, respectively.
(F_2)

$$\lim_{|(u,v)|\to 0} \frac{|F(x,u,v)|}{M_1(u)+M_2(v)} = 0, \qquad \text{uniformly for all } x \text{ in } \Omega. \tag{I.11}$$

And

$$\lim_{|(u,v)|\to+\infty} \frac{F(x,u,v)}{M_1(u)+M_2(v)} = \infty, \qquad \text{uniformly for all } x \text{ in } \Omega. \tag{I.12}$$

(F_3): There exists a continuous function $\gamma : [0,\infty) \to \mathbb{R}$ such that

$$1 < l_\Gamma := \inf_{t>0} \frac{t\gamma(t)}{\Gamma(t)} \leq \sup_{t>0} \frac{t\gamma(t)}{\Gamma(t)} := n_\Gamma < +\infty, \tag{I.13}$$

where

$$\Gamma(t) := \int_0^{|t|} \gamma(r)dr, \quad \text{for all } t \in \mathbb{R},$$

is an N-function and functions $H_i(t) := |t|^{\frac{l_i l_\Gamma}{l_\Gamma - 1}}, t \in \mathbb{R}$ increase essentially more slowly than M_i near infinity, respectively, such that

$$\Gamma\Big(\frac{F(x,u,v)}{|u|^{l_1}+|v|^{l_2}}\Big) \leq c_2 \overline{F}(x,u,v), \quad x \in \mathbb{R}, \quad |(u,v)| \geq r, \tag{I.14}$$

where c_2 and r are tow strictly positive constants and

$$\overline{F}(x,u,v) := \frac{1}{n_1} F_u(x,u,v)u + \frac{1}{n_2} F_v(x,u,v)v - F(x,u,v),$$

$\forall (x,u,v) \in \Omega \times \mathbb{R} \times \mathbb{R}$.

A. Examples

We set some examples which are in at least one M and $\overline{M}$ can be not reflexive:
1) $m(t) = |t|^{p-1}$ where $1 < p < \infty$. This is a case of polynomial growth, M and $\overline{M}$ satisfy the Δ_2-condition. We are in a reflexive situation, the classical theory of monotone operators in reflexive Banach can be applied.
2) $m(t) = \text{sgn}\, t \log(1+|t|)$ This is a case of slow growth, M satisfies the Δ_2-condition but $\overline{M}$ does not.
3) $m(t) = \text{sgn}\, t.(e^{|t|}-1)$. This is a case of rapid growth, M does not satisfy the Δ_2-condition but $\overline{M}$ does. For further examples we refer to ([11] p 28).

This paper is organized as follows: In the second Section, we recall some well-known properties and results on Orlicz and Orlicz Sobolev spaces. Third Section we present the existence of a solution to the problem (I.1) and its proof which relies on the Mountain Pass Theorem.

II. Some preliminary results and Hypotheses

In this section, we list some basic properties of the Orlicz-Sobolev Space. We refer the reader to [8], [10], [11] for further references and for some of the proofs of the results in this section.

Let Ω be an open subset of $\mathbb{R}^N$, $N \in \mathbb{N}$.

Definition 2.1: The N-function M satisfies a Δ_2-condition globally, if for some constant $k > 2$,

$$M(2t) \leq k\, M(t), \quad \text{for every } t > 0. \tag{II.1}$$

The Orlicz space $L_M(\Omega)$ is defined as the set of equivalence classes of real-valued measurable functions u on Ω such that:

$$\int_\Omega M\Big(\frac{u(x)}{\lambda}\Big)\, dx < +\infty \text{ for some } \lambda > 0. \tag{II.2}$$

Notice that $L_M(\Omega)$ is a Banach space under the so-called Luxemburg norm, namely

$$\|u\|_M = \inf\Big\{\lambda > 0 \,/ \int_\Omega M\Big(\frac{u(x)}{\lambda}\Big)\, dx \leq 1\Big\}. \tag{II.3}$$

In $L_M(\Omega)$ we define the Orlicz norm $||u||_{(M)}$ by

$$||u||_{(M)} = \sup \int_\Omega u(x)v(x)\, dx, \tag{II.4}$$

where the supremum is taken over all $v \in E_{\overline{M}(\Omega)}$ such that $||v||_{\overline{M},\Omega} \leq 1$. An important inequality in $L_M(\Omega)$ is the following:

$$\int_\Omega M(u(x))\, dx \leq ||u||_{(M)} \text{ for all } u \in L_M(\Omega) \text{ such that } ||u||_{(M)} \leq 1, \tag{II.5}$$

wherefrom we readily deduce

$$\int_\Omega M\Big(\frac{u(x)}{||u||_{(M)}}\Big)\, dx \leq 1 \text{ for all } u \in L_M(\Omega) \setminus \{0\}. \tag{II.6}$$

It can be shown that the norm $||\cdot||_{(M)}$ is equivalent to the Luxemburg norm $\|\cdot\|_{M,\Omega}$. Indeed,

$$||u||_{M,\Omega} \leq ||u||_{(M)} \leq 2||u||_{M,\Omega} \text{ for all } u \in L_M(\Omega). \tag{II.7}$$

We have the following inequality

$$||u||_{(M),\Omega} \leq \int_\Omega M(u(x))dx + 1 \text{ for all } u \in L_M(\Omega). \tag{II.8}$$

Also, the Hölder inequality holds

$$\int_\Omega |u(x)v(x)|\, dx \leq ||u||_{M,\Omega}||v||_{(\overline{M})} \text{ for all } u \in L_M(\Omega) \text{ and } v \in L_{\overline{M}}(\Omega),$$

in particular, if Ω has finite measure, Hölder's inequality yields the continuous inclusion $L_M(\Omega) \subset L^1(\Omega)$.
For Orlicz spaces Young inequality reads as follows:

$$st \leq M(s) + \overline{M}(t) \ \ \textit{for all } t,\ s \geq 0 \textit{ and } x \in \Omega. \tag{II.9}$$

Lemma 2.2: ([10]) If $M_2 \ll M_1$ then

$$L_{M_1}(\Omega) \hookrightarrow L_{M_2}(\Omega)$$

We now turn to the Orlicz-Sobolev Space defined by

$$W^1L_M(\Omega) := \Big\{u \in L_M(\Omega) : \frac{\partial u}{\partial x_i} \in L_M(\Omega), i = 1, ..., N\Big\}$$

equipped with the norm $||u||_{1,M} = ||u||_M + ||\nabla u||_M$ and

$$W^1E_M(\Omega) := \Big\{u \in E_M(\Omega) : \frac{\partial u}{\partial x_i} \in E_M(\Omega), i = 1, ..., N\Big\}$$

Thus $W^1L_M(\Omega)$ and $W^1E_M(\Omega)$ are tow Banach spaces under the Luxemburg norm. Denote

$$W_0^1L_M(\Omega) = \overline{C^\infty}(\Omega)^{\|\cdot\|_{1,M}}.$$

Definition 2.3: Let $(u_k) \in L_M(\Omega)$ and $u \in L_M(\Omega)$. We say that u_k converges to u for the modular convergence in $L_M(\Omega)$ if for some $\lambda > 0$, $\int_\Omega M\Big(\frac{u_k - u}{\lambda}\Big)\,dx \to 0$. The fact that M satisfies a Δ_2-condition globally implies that

$$u_k \to u \quad in \quad L_M(\Omega) \Leftrightarrow \int_\Omega M((u_k - u))dx \to 0. \qquad \text{(II.10)}$$

Theorem 2.4 (Generalized Poincaré Inequality): Let Ω be a bounded open subset of $\mathbb{R}^N$ and let M be an N-function. Then there exists a positive constant μ such that,

$$||u||_M \leq \mu ||u||_{0,M}, \qquad \forall u \in W_0^1 L_M(\Omega). \qquad \text{(II.11)}$$

Notation : In this work we note $W_0^1 L_M(\Omega)$ by $W_0^{1,M}(\Omega)$ and $W^1 L_M(\Omega)$ by $W^{1,M}(\Omega)$
Next, we give some inequalities which will be used in our proofs. For the proofs, we refer the reader to the papers [1], [9].

Lemma 2.5: Let $\xi_0(t) = \min\{t^l, t^n\}$, $\xi_1(t) = \max\{t^l, t^n\}$, $t \geq 0$, M is an N-function, then the following conditions are equivalent :

1)

$$1 < l := \inf_{t>0} \frac{tm(t)}{M(t)} \leq \sup_{t>0} \frac{tm(t)}{M(t)} := n < N. \qquad \text{(II.12)}$$

2)

$$\xi_0(t)M(\rho) \leq M(\rho t) \leq \xi_1(t) M(\rho), \qquad \forall t, \rho \geq 0.$$

3) M satisfies a Δ_2-condition globally.

Lemma 2.6: If (II.12) hold then

$$\xi_0(||u||_{M,\Omega}) \leq \int_\Omega M(u)dx \leq \xi_1(||u||_{M,\Omega}), \qquad \forall u \in L_M(\Omega).$$

Lemma 2.7: Let $\overline{M}$ be the complement of M and $\xi_2(t) = \min\{t^{\overline{l}}, t^{\overline{n}}\}$, $\xi_3(t) = \max\{t^{\overline{l}}, t^{\overline{n}}\}$, $t \geq 0$ where $\overline{l} = \frac{l}{l-1}$ and $\overline{n} = \frac{n}{n-1}$, If M is an N-function and (II.12) holds with $l > 1$. then $\overline{M}$ satisfies:

1)

$$\overline{n} = \inf_{t>0} \frac{t\overline{M}'(t)}{\overline{M}(t)} \leq \sup_{t>0} \frac{t\overline{M}'(t)}{\overline{M}(t)} = \overline{l}.$$

2)

$$\xi_2(t)\overline{M}(\rho) \leq \overline{M}(\rho t) \leq \xi_3(t)\overline{M}(\rho), \qquad \forall t, \rho \geq 0.$$

3)

$$\xi_2(||u||_{\overline{M}}) \leq \int_\Omega \overline{M}(u)dx \leq \xi_3(||u||_{\overline{M}}), \qquad \forall u \in L_{\overline{M}}(\Omega).$$

Lemma 2.8: Let $\xi_4(t) = \min\{t^{l^*}, t^{n^*}\}$, $\xi_5(t) = \max\{t^{l^*}, t^{n^*}\}$, $t \geq 0$ where $l^* = \frac{lN}{N-l}$ and $n^* = \frac{nN}{N-n}$. If M is an N-function and (II.12) in Lemma 2.7 hold with $l, n \in (1, N)$, then M^* satisfies:

1)

$$l^* = \inf_{t>0} \frac{t(M^*)'(t)}{M^*(t)} \leq \sup_{t>0} \frac{t(M^*)'(t)}{M^*(t)} = n^*.$$

2)

$$\xi_4(t)M^*(\rho) \leq M^*(\rho t) \leq \xi_5(t)M^*(\rho), \qquad \forall t, \rho \geq 0.$$

3)

$$\xi_4(||u||_{M^*}) \leq \int_\Omega M^*(u)dx \leq \xi_5(||u||_{M^*}), \qquad \forall u \in L_{M^*}(\Omega).$$

Lemma 2.9: Under the assumption of Lemma 2.8, the embedding from $W_0^{1,M}(\Omega)$ into $L_{M^*}(\Omega)$ is continuous and into $L_\Phi(\Omega)$ is compact for any N-function Φ increasing essentially more slowly than M^* near infinity.

Lemma 2.10: M increases essentially more slowly than M^* near infinity,
i.e,

$$\lim_{t\to\infty} \frac{M(kt)}{M^*(t)} = 0 \qquad for \quad every \quad constant\, k > 0.$$

Proof 2.11: by 2) Lemma 2.5 and 2) Lemma 2.8,

$$0 \leq \frac{M(kt)}{M^*(t)} \leq \frac{M(k)\xi_1(t)}{M^*(1)\xi_2(t)} = \frac{M(k)t^n}{M^*(1)t^{l^*}}$$

for $1 \leq t$. Since $n < l^*$, we have the result.

Due to the nature of M-Laplacian operator defined in I.2 we need to consider the Orlicz-Sobolev framework and we will examine some specific techniques to Orlicz and the Orlicz-Sobolev spaces. For that we define

$$W := W_0^{1,M_1}(\Omega) \times W_0^{1,M_2}(\Omega)$$

equipped with the following norm $||u, v|| = ||\nabla u||_{M_1} + ||\nabla v||_{M_2}$.
We can see that W is a separable and reflexive Banach space.

Definition 2.12: We define a weak solution (u, v) in W to the problem (I.1) by

$$\langle -\Delta_{m_1} u, \overline{u}\rangle + \langle -\Delta_{m_2} v, \overline{v}\rangle = \int_\Omega F_u(x, u, v)\overline{u}dx + \int_\Omega F_v(x, u, v)\overline{v}dx \qquad \text{(II.13)}$$

for all $(\overline{u}, \overline{v}) \in W$, where

$$\langle -\Delta_{m_1} u, \overline{u}\rangle = \langle \mathcal{H}_1(u), \overline{u}\rangle := \int_\Omega m_1(|\nabla u|)\nabla u \nabla \overline{u}dx,$$

and

$$\langle -\Delta_{m_2} v, \overline{v}\rangle = \langle \mathcal{H}_2(v), \overline{v}\rangle := \int_\Omega m_2(|\nabla v|)\nabla v \nabla \overline{v}dx.$$

Now define the operators $\mathcal{H}_i : W_0^{1,M_i}(\Omega) \to \Big(W_0^{1,M_i}(\Omega)\Big)^*$ by

$$\langle \mathcal{H}_i(u), \overline{u}\rangle := \int_\Omega m_i(|\nabla u|)\nabla u \nabla \overline{u}dx.$$

Lemma 2.13: [5] The function $\mathcal{H}_i$ is of type $(S+)$. i,e. given a sequence (u_k) converges weakly to u in $W_0^{1,M_i}(\Omega)$ and

$$\lim_{n\to\infty} \sup \langle \mathcal{H}_i(u_k), u_k - u\rangle \leq 0. \qquad \text{(II.14)}$$

Then (u_k) converge strongly to $u \in W_0^{1,M_i}(\Omega)$.

We observe that the energy functional I on W corresponding to system (I.1) is

$$\begin{aligned} I(u,v) &:= \int_\Omega M_1(|\nabla u|)dx + \int_\Omega M_2(|\nabla v|) \\ &\quad - \int_\Omega F(x,u,v)dx, \end{aligned}$$

$for\ \ all \quad (u,v) \in W$. Denote by $I_i (i = 1,2) : W \to \mathbb{R}$ the functionals

$$I_1(u,v) := \int_\Omega M_1(|\nabla u|)dx + \int_\Omega M_2(|\nabla v|)dx$$

and

$$I_2(u,v) = \int_\Omega F(x,u,v)dx.$$

Then $I(u,v) = I_1(u,v) - I_2(u,v)$.
The function I_1 is well-defined and of class $C^1(W,\mathbb{R})$ and we have the following representation

$$\begin{aligned} &\langle I'(u,v),(\overline{u},\overline{v})\rangle = \langle \mathcal{H}_1(u),\overline{u}\rangle + \langle \mathcal{H}_2(v),\overline{v}\rangle \\ &- \int_\Omega F_u(x,u,v)\overline{u}dx - \int_\Omega F_v(x,u,v)\overline{v}dx, \end{aligned}$$

for all $(\overline{u},\overline{v}) \in W$. Then, the critical points of I on W are weak solutions of system I.1.

III. MAIN RESULTS

In this section, we present the following existence result by using mountain pass theorem, see [14].

Theorem 3.1: Assume that $(F_0)-(F_2)$ and (F_3) hold. Then system (I.1) possesses a nontrivial weak solution.

Remark 3.2: Under assumptions (F_1) and (F_3), by Lemma 2.10, the following embeddings $W_0^{s,m_i}(\Omega) \to L^{\Psi_i}(\Omega)$, $W_0^{s,m_i}(\Omega) \to L^{l_i \tilde{l}_\Gamma}(\Omega)$ and $W_0^{s,m_i}(\Omega) \to L^{l_i \tilde{m}_\Gamma}(\Omega)$ are compact where $\tilde{l} = \frac{l_\Gamma}{l_\Gamma - 1}$ and $\tilde{m} = \frac{m_\Gamma}{m_\Gamma - 1}$.

Remark 3.3: By 2) in Lemma 2.5, assumptions (F_2) and (F_3) show

$$\lim_{|(u,v)|\to+\infty} \tilde{F}(x,u,v) \to +\infty, \qquad uniformly\ for\ all\ \ x \in \Omega.$$

Remark 3.4: Based on the Youngs inequality (II.9), $F(x,0,0) = 0$ and the fact

$$F(x,u,v) = \int_0^u F_s(x,s,0)ds + \int_0^v F_t(x,0,t)dt + F(x,0,0),$$

$\forall (x,u,v) \in \Omega \times \mathbb{R} \times \mathbb{R}$.
By (I.10) and 2) in Lemma 2.5, show that there exists a constant $c_4 > 0$ such that

$$|F(x,u,v)| \le c_4(\Psi_1(u) + \Psi_2(v)), \quad \forall (x,u,v) \in \Omega \times \mathbb{R} \times \mathbb{R}. \tag{III.1}$$

Theorem 3.5: Let E be a real Banach space with its dual space E^*, and suppose that $J \in C^1(E,\mathbb{R})$ satisfies

$$max\{J(0),J(e)\} \le \alpha < \beta \le \inf_{||u||=\rho} J(u),$$

for some $\alpha > \beta$, $\rho > 0$ and $e \in E$ with $||e|| > \rho$. Let $c \ge \beta$ be characterized by

$$c = \inf_{\gamma\in\Gamma} max_{t\in[0,1]} J(\gamma(t))$$

where $\Gamma = \{\gamma \in C([0,1],E) : \gamma(0) = 0, \gamma(1) = e\}$ is the set of continuous paths joining 0 and e, then there exists a sequence $\{u_k\} \subset E$ such that

$$J(u_k) \to c \ge \beta \ \ and \ \ ||J'(u_k)||_{E^*}(1+||u_k||) \to 0 \quad as \quad n \to \infty. \tag{III.2}$$

This kind of sequence is usually called a Cerami sequence.

Definition 3.6: Let $J : W_0^{1,M}(\Omega) \to \mathbb{R}$ is a class C^1. We say that a sequence u_k in a Banach Space $W_0^{1,M}(\Omega)$ is a Cerami sequence (in short $(C)_c$) at the level $c \in \mathbb{R}$ for the functional J when

$$J(u_k) \to c \quad and \quad (1 + \|u_k\|)\|J'(u_k)\| \to 0.$$

Lemma 3.7: Let E be a real Banach Space and $I \in C^1(E,R)$ satisfying (PS)-condition. Suppose $I(0) = 0$ and
(I_1) there are constants $\rho, \alpha > 0$ such that $I|_{\partial B_\rho} \ge \alpha$.
(I_2) there is an $e \in E \backslash B_\rho$ such that $I(e) \le 0$.
Then I possesses a critical value $c \ge \alpha$.

Lemma 3.8: Suppose that (F_1) hold. Then there are constants $\rho, \alpha > 0$ such that $I|_{\partial B_\rho} \ge \alpha$.

Proof 3.9: By equation (I.10) there exists $c_4 > 0$ such that

$$|F(x,u,v)| \le (1-\epsilon)(\lambda_1 M_1(u) + \lambda_2 M_2(v)) + c_4(1 + \Psi_1(u) + \Psi_2(v)),$$

where $||u,v|| \le 1$, by Poincaré's Inequality and Lemma 2.5 we obtain

$$\begin{aligned} I(u,v) = &\int_\Omega M_1(|\nabla u|))dx + \int_\Omega M_2(|\nabla v|)dx - \int_\Omega F(x,u,v)dx \\ \ge\ & \epsilon \min\{||\nabla u||_{M_1}^{l_1}, ||\nabla u||_{M_1}^{n_1}\} + \epsilon \min\{||\nabla v||_{M_2}^{l_2}, ||\nabla v||_{M_2}^{n_2}\} \\ &-c_4 \int_\Omega \Psi_1(u)dx - c_4 \int_\Omega \Psi_2(v)dx \\ \ge\ & ||\nabla u||_{M_1}^{n_1}(\epsilon - c_4||\nabla u||_{M_1}^{l_{\Psi_1 - n_1}}) \\ &+||\nabla v||_{M_2}^{n_2}(\epsilon - c_4||\nabla v||_{M_2}^{l_{\Psi_2 - n_2}}), \end{aligned}$$

since $1 < n_i < l_{\Psi_i}$ we can choose positive constants ρ and α small enough such that $I(u,v) > \alpha$ for all $(u,v) \in W$ with $||(u,v)|| = \rho$.

Lemma 3.10: Suppose that (F_3) hold. Then there is a point $(u,v) \in W \setminus B_\rho$ such that $I(u,v) \le 0$.

Proof 3.11: By (F_3) and the fact that F is continuous, then for any given constant $G > 0$, there exists a constant $C_G > 0$ such that

$$F(x,u,v) \ge G(M_1(u) + M_2(v)) - C_G \qquad \forall (x,u,v) \in \Omega \times \mathbb{R} \times \mathbb{R}. \tag{III.3}$$

Now, choose $u_0 \in C_c^1(\Omega) \setminus \{0\}$ with $0 \le u_0(x) \le 1$. Then $(u_0, 0) \in W$, and by (III.3) and 2) in Lemma 2.5, when $t > 0$ we have

$$\begin{aligned} I(tu_0,0) &= \int_\Omega M_1(t|\nabla u_0|)d\mu - \int_\Omega F(x,tu_0,0)dx \\ &\le M_1(t)\big(|||\nabla u_0|||_{l_1}^{l_1} + |||\nabla u_0|||_{n_1}^{n_1} - M||u_0||_{n_1}^{n_1}\big) + C_G|\Omega|, \end{aligned}$$

Since $G > 0$ is arbitrary and $\lim_{t\to\infty} M_1(t) = +\infty$, we can choose $G > \dfrac{|||\nabla u_0|||_{l_1}^{l_1} + |||\nabla u_0|||_{n_1}^{n_1}}{||u_0||_{n_1}^{n_1}}$ and large t such that $I(tu_0,0) \le 0$ and $||(tu_0,0)|| > \rho$.

Lemma 3.12: Suppose that $(F_0)-(F_2)$ and (F_3) hold. Then $(C)_c$-sequence in W is bounded.

Proof 3.13: Let $\{u_k, v_k\} \in W$ be a $(C)_c$-sequence of I in W, then for n large enough by (III.2), we obtain

$$c+1 \geq I(u_k,v_k) - \langle I'(u_k,v_k), (\frac{1}{n_1}u_k, \frac{1}{n_2}v_k)\rangle$$

$$= \int_\Omega M_1(|\nabla u_k|)dx + \int_\Omega M_2(|\nabla v_k|)dx - \int_\Omega F(x,u_k,v_k)dx$$

$$- \frac{1}{n_1}\int_\Omega m_1(|\nabla u_k|)|\nabla u_k|^2 dx - \frac{1}{n_2}\int_\Omega m_2(|\nabla v_k|)|\nabla v_k|^2 dx$$

$$\geq \int_\Omega \overline{F}(x,u_k,v_k)dx,$$

by contradiction, we prove the boundedness of sequence $\{(u_k,v_k)\}$. Suppose that there exists a sub-sequence of $\{(u_k,v_k)\}$, still denoted by $\{(u_k,v_k)\}$, such that

$$||(u_k,v_k)|| = ||\nabla u_k||_{M_1} + ||\nabla v_k||_{M_2} \to +\infty.$$

Next, we discuss the problem in two cases.

Case1: Suppose that $||\nabla u_k||_{M_1} \to +\infty$ and $||\nabla v_k||_{M_2} \to +\infty$. Let $\overline{u}_k = \frac{u_k}{||\nabla u_k||_{M_1}}$ and $\overline{v}_k = \frac{v_k}{||\nabla v_k||_{M_2}}$. Then $\{(\overline{u}_k,\overline{v}_k)\}$ is bounded in separable, reflexive Banach space W. Passing to a subsequence less denoted by $\{(\overline{u}_k,\overline{v}_k)\}$ by Remark 3.2, there exists a point $(\overline{u},\overline{v}) \in W$ such that:

(a) $\overline{u}_k \rightharpoonup \overline{u}$ in $W_0^{1,M_1}(\Omega)$; $\overline{u}_k \to \overline{u}$ in $L^{l_1\bar{l}_\Gamma}(\Omega)$ and $L^{l_2\overline{n}_\Gamma}(\Omega)$; $\overline{u}_k \to \overline{u}$ in a.e in Ω.

(b) $\overline{v}_k \rightharpoonup \overline{v}$ in $W_0^{1,M_2}(\Omega)$; $\overline{v}_k \to \overline{v}$ in $L^{l_2\bar{l}_\Gamma}(\Omega)$ and $L^{l_2\overline{n}_\Gamma}(\Omega)$; $\overline{v}_n \to \overline{v}$ in a.e in Ω.

Firstly, we assume that $[\overline{u} \neq 0] := [x \in \Omega : \overline{u}(x) \neq 0]$ or $[\overline{v} \neq 0] := [x \in \Omega : \overline{v}(x) \neq 0]$ has nonzero Lebesgue measure. It is clear that

$$|u_k| = |\overline{u}_k|\,||\nabla u_k||_{M_1} \to +\infty \qquad in\ [\overline{u} \neq 0],$$

and

$$|v_k| = |\overline{v}_k|\,||\nabla u_k||_{M_1} \to +\infty \qquad in\ [\overline{v} \neq 0].$$

Then, by (III.4) and Fatou's Lemma, we have

$$c+1 \geq \int_\Omega \overline{F}(x,u_k,v_k)dx \to +\infty,$$

which is a contradiction. Next, we assume that both $[\overline{u} \neq 0]$ and $[\overline{v} \neq 0]$ have zero Lebesgue measure, that is $\overline{u} = 0$ in $W_0^{1,M_1}(\Omega)$ and $\overline{v} = 0$ in $W_0^{1,M_2}(\Omega)$. By Lemma 2.6, we have

$$\begin{aligned} &\min\{||\nabla u_k||^{l_1}_{M_1}, ||\nabla u_k||^{n_1}_{M_1}\} \\ &+\min\{||\nabla u_k||^{l_2}_{M_2}, ||\nabla v_k||^{n_2}_{M_2}\} \\ &\leq I(u_k,v_k) + \int_\Omega F(x,u_k,v_k)dx. \end{aligned} \tag{III.5}$$

When k is large enough, that is

$$||\nabla u_k||^{l_1}_{M_1} + ||\nabla u_k||^{l_2}_{M_2} \leq I(u_k,v_k) + \int_\Omega F(x,u_k,v_k)dx,$$

which is equivalent to

$$\begin{aligned} 1 \leq\ & \frac{I(u_k,v_k)}{||\nabla u_k||^{l_1}_{M_1} + ||\nabla u_k||^{l_2}_{M_2}} + \int_{|u_k,v_k|\leq R} \frac{F(x,u_k,v_k)}{||\nabla u_k||^{l_1}_{M_1} + ||\nabla v_k||^{l_2}_{M_2}}dx \\ +& \int_{|u_k,v_k|>R} \frac{F(x,u_k,v_k)}{||\nabla u_k||^{l_1}_{M_1} + ||\nabla v_k||^{l_2}_{M_2}}dx = o_k(1) \\ +& \int_{|u_k,v_k|\leq R} \frac{F(x,u_k,v_k)}{||\nabla u_k||^{l_1}_{M_1} + ||\nabla u_k||^{l_2}_{M_2}}dx \\ +& \int_{|u_k,v_k|>R} \frac{F(x,u_k,v_k)}{||\nabla u_k||^{l_1}_{M_1} + ||\nabla v_k||^{l_2}_{M_2}}dx, \end{aligned} \tag{III.4}$$

where R is a positive constant such that $R > r$ (see (F_4)), bearing in mind that $|(u,v)| > R$ and by (F_2) we have

$$F(x,u,v) \geq 0, \qquad x \in \Omega \qquad .$$

For $|(u,v)| \leq R$ and the fact that F is continuous, there exists a constant $C_R > 0$ such that

$$|F(x,u,v)| < C_R, \qquad \forall x \in \Omega, \tag{III.7}$$

then

$$\int_{|u_k,v_k|\leq R} \frac{F(x,u_k,v_k)}{||\nabla u_k||^{l_1}_{M_1} + ||\nabla v_k||^{l_2}_{M_2}}dx \leq o_k(1). \tag{III.8}$$

Besides, it follows from Höder's inequality that

$$\begin{aligned} &\int_{|u_k,v_k|>R} \frac{F(x,u_k,v_k)}{||\nabla u_k||^{l_1}_{M_1} + ||\nabla v_k||^{l_2}_{M_2}}dx \\ &\leq 2\left\|\frac{F(x,u_k,v_k)}{|u_k|^{l_1}+|v_k|^{l_2}}\chi_{\{|(u_k,v_k)|>R\}}\right\|_\Gamma \\ &\times||(|\overline{u}_n|^{l_1}+|\overline{v}_n|^{l_2})\chi_{\{|(u_k,v_k)|>R\}}||_{\overline{\Gamma}}, \end{aligned} \tag{III.9}$$

where χ denotes the characteristic function which satisfies

$$\chi_{\{|(u_k(x),v_k(x))|>R\}} = \begin{cases} 1 & for\ x \in \{x \in \Omega : |(u_k(x),v_k(x))| > R\} \\ 0 & for\ x \in \{x \in \Omega : |(u_k(x),v_k(x))| \leq R\} \end{cases}$$

For k large enough, by (I.14), (III.4) and the fact that $\overline{F}$ is continuous, we obtain

$$\int_\Omega \Gamma\Big(\frac{F(x,u_k,v_k}{|u_k|^{l_1}+|v_k|^{l_2}}\chi_{\{|(u_k,v_k)|>R\}}\Big)dx$$

$$\leq c_2\int_\Omega \overline{F}(x,u_k,v_k)dx + C \leq c_2(c+1)+C$$

Then, for k large enough, by Lemma 2.6, there exists a constant $c_6 > 0$ such that

$$\left\|\frac{F(x,u_k,v_k}{|u_k|^{l_1}+|v_k|^{l_2}}\chi_{\{|(u_k,v_k)|>R\}}\right\|_\Gamma \leq c_6. \tag{III.10}$$

Moreover, it is easy to see that

$$||(|\overline{u}_k|^{l_1}+|\overline{v}_k|^{l_2})\chi_{\{|(u_k,v_k)|>R\}}||_{\overline{\Gamma}} \leq \|\,|\overline{u}_k|^{l_1}$$

$$+|\overline{v}_k|^{l_2}||_{\overline{\Gamma}} \leq \|\,|\overline{u}_k|^{l_1}||_{\overline{\Gamma}} + \|\,|\overline{v}_k|^{l_2}||_{\overline{\Gamma}}.$$

By Lemma 2.5, Lemma 2.7 and (F_3) implies that N-function $\overline{\Gamma}$ satisfies a Δ_2-condition globally. Then by (II.10), $||u||_{\overline{\Gamma}} \to 0$ as $\int_\Omega \overline{\Gamma}(|u|)dx \to 0$. It follows from Lemma 2.7 , (a) and (b) in case 1 that

$$\int_\Omega \overline{\Gamma}(|\overline{u}_k|^{l_1})dx + \int_\Omega \overline{\Gamma}(|\overline{v}_k|^{l_2})dx \quad \leq \quad o_k(1),$$

which implies

$$||(|\overline{u}_k|^{l_1} + |\overline{v}_k|^{l_2})\chi_{\{|(u_k,v_k)|>R\}}||_{\overline{\Gamma}} \leq \|\, |\overline{u}_k|^{l_1}||_{\overline{\Gamma}} + \|\, |\overline{v}_k|^{l_2}||_{\overline{\Gamma}} \tag{III.11}$$

$= o_k(1)$.
By combining (III.8), (III.9), (III.10), (III.11) with (III.6) we get a contradiction.

Case2. Suppose that $||\nabla u_k||_{M_1} \leq C$ or $||\nabla v_k||_{M_2} \leq C$, for some $C > 0$ and all $k \in \mathbb{N}$. Without loss of generality, we assume that $||\nabla u_k||_{M_1} \to +\infty$ and $||\nabla v_k||_{M_2} \leq C$, for some $C > 0$ and for all $k \in \mathbb{N}$. Let $\overline{u}_k = \dfrac{u_k}{||\nabla u_k||_{M_1}}$ and $\overline{v}_k = \dfrac{v_k}{||\nabla u_k||_{M_1}}$ then $||\overline{v}_k||_{0,M_2} \to 0$ and $||\overline{u}_k||_{0,M_1} \to 1$. By Remark 3.2, there exists a point $(\overline{u}, \overline{v}) \in W$ such that:
(c) $\overline{u}_k \rightharpoonup \overline{u}$ in $W_0^{1,M_1}(\Omega)$, $\overline{u}_k \to \overline{u}$ in $L^{l_1\bar{l}_\Gamma}(\Omega)$ and $L^{l_2\overline{n}_\Gamma}(\Omega)$ $\overline{u}_k \to \overline{u}$ in a.e in Ω,
(d) $\overline{v}_k \rightharpoonup \overline{v}$ in $W_0^{1,M_2}(\Omega)$; $\overline{v}_k \to \overline{v}$ in $L^{l_2\bar{l}_\Gamma}(\Omega)$ and $L^{l_2\overline{n}_\Gamma}(\Omega)$; $\overline{v}_k \to \overline{v}$ in a.e in Ω. Similarly, we firstly assume that $[\overline{u} \neq 0\ ]$ has nonzero Lebesgue measure. We can see that

$$|u_k| = |\overline{u}_k|\, ||u_k||_{s,M_1} \to +\infty, \qquad\qquad in\ [\overline{u} \neq 0].$$

Then, by (III.4) and Fatou's Lemma, we get a contradiction by

$$c + 1 \geq \int_\Omega \overline{F}(x, u_k, v_k)dx \to +\infty.$$

Next, we suppose that $[\overline{u} \neq 0]$ has zero Lebesgue measure, that is $\overline{u} = 0$ in $W_0^{1,M_1}(\Omega)$. By Lemma 2.7 and (c) and (d) in case 2 we have

$$\min\left\{ \|\, |v_k|^{l_2}\|_{\overline{\Gamma}}^{\bar{l}_\Gamma}, \|\, |v_k|^{l_2}\|_{\overline{\Gamma}}^{\overline{n}_\Gamma} \right\} \quad \leq \quad \overline{\Gamma}(1) \times \left(\int_\Omega |v_k|^{l_2\bar{l}_\Gamma}dx + \int_\Omega |v_k|^{l_2\overline{n}_\Gamma} \right)dx \to C,$$

Then there existe a constant $L > 0$ such that

$$\|\, |v_k|^{l_2}\|_{\overline{\Gamma}} \leq L, \qquad \forall k \in \mathbb{N}. \tag{III.12}$$

When k large enough, (III.5) changed into

$$||\nabla u_k||_{M_1}^{l_1} + K \leq I(u_k, v_k) + \int_\Omega F(x, u_k, v_k) + K,$$

where K is a positive constant with $K > 4Lc_6$ (see (III.10) and (III.12)). Then by (III.7) , (III.10), (III.11) , (III.12) and Höder's Inequality, above estimate means

$$\begin{aligned} 1 \quad &\leq \quad \frac{I(u_k, v_k) + K}{||\nabla u_k||_{M_1}^{l_1} + K} + \int_\Omega \frac{F(x, u_k, v_k)}{||\nabla u_k||_{M_1}^{l_1} + K}dx \\ &\leq \quad o_k(1) + 2c_6\big(o_k(1) + \frac{L}{K}\big) < o_k(1) + \frac{1}{2}, \end{aligned}$$

which is a contradiction.

Lemma 3.14: Suppose that $(F_1)-(F_3)$ hold. Then I satisfies $(C)_c$-condition.

Proof 3.15: Let $\{(u_k, v_k)\}$ be any $(C)_c$-sequence of I in W. Lemma 2.13 shows $\{(u_k, v_k)\}$ is bounded. Passing to a subsequence denote by $\{(u_k, v_k)\}$, there exists a point $(u, v) \in W$ such that:
(e) $u_k \rightharpoonup u$ in $W_0^{1,M_1}(\Omega)$, $u_k \to u$ in $L^{\Psi_1}(\Omega)$, $u_k \to u$ a.e Ω.
(f) $v_k \rightharpoonup v$ in $W_0^{1,M_2}(\Omega)$, $v_k \to v$ in $L^{\Psi_2}(\Omega)$, $v_k \to v$ a.e Ω. then we have

$$\begin{aligned} \langle \mathcal{H}_1(u_k), u_k - u \rangle &= \int_\Omega m_1(|\nabla u_k|)\nabla u_k \nabla(u_k - u)\, dx \qquad \text{(III.13)} \\ &= \langle I'(u_k, v_k), (u_k - u, 0) \rangle + \int_\Omega F_u(x, u_k, v_k)dx. \end{aligned}$$

Equation (III.2) shows that

$$\Big|\langle I'(u_k, v_k), (u_k - u, 0) \rangle\Big| \leq \|I'(u_k, v_k)\|_{W_0^{-1,\overline{M_1}}} \|u_k - u\|_{W_0^{1,M_1}} \to 0. \tag{III.14}$$

By (F_1) and Höder's inequality, we get

$$\begin{aligned} &\left| \int_\Omega F_u(x, u_k, v_k)(u_k - u)dx \right| \qquad \text{(III.15)} \\ &\leq 2c_1||1 + \psi_1(|u_k|) + \overline{\Psi}_1^{-1}(\Psi_2(|v_k|))||_{\overline{\Psi}_1} ||u_k - u||_{\Psi_1}. \end{aligned}$$

Condition (F_1) shows that functions Ψ_1 and $\overline{\Psi}_1$ are N-functions satisfying Δ_2-condition globally, which together with the convexity of N-function, Lemma 2.6 and the boundedness of $\{(u_k, v_k)\}$, imply that

$$\begin{aligned} &\int_\Omega \overline{\Psi}_1(1 + \psi_1(|u_k|)) + \overline{\Psi}_1^{-1}(\Psi_2(|v_k|))dx \\ &\leq C \int_\Omega (\Psi_1(u_k) + \Psi_2(v_k))dx \leq C, \end{aligned}$$

which, together with Lemma 2.6 again, shows that

$$||1 + \psi_1(|u_k|) + \overline{\Psi}_1^{-1}(\Psi_2(|v_k|))||_{\overline{\Psi}_1} \leq C, \tag{III.16}$$

for some $C > 0$. Moreover, (e) and (f) shows that

$$||u_k - u||_{\Psi_1} \to 0. \tag{III.17}$$

then, combining (III.13), (III.14), (III.15), (III.16) and (III.17) we obtain

$$\langle \mathcal{H}_1, u_k - u \rangle \to 0 \ \ as \ \ n \to \infty.$$

or $\mathcal{H}$ is of the class (S+), then $u_k \to u$ in $W^{1,M_1}(\Omega)$ and $v_k \to v$ in $W^{1,M_2}(\Omega)$ Therefore $(u_k, v_k) \to (u, v)$ in W.

*Proof 3.16 (**Proof of theorem 3.1**):* By Lemmas 3.10 , 2.13 , 3.14 and the obvious fact $I(0) = 0$, all conditions of Lemma 3.7 hold. Then system (I.1) possesses a nontrivial weak solution which is a critical point of I.

References

[1] R. A. Adams, J. F. Fournier, Sobolev spaces, Second edition, Pure and Applied Mathematics (Amsterdam), Elsevier/Academic Press, Amsterdam, (2003).

[2] K. Adriouch, A. El Hamidi, The Nehari manifold for systems of nonlinear elliptic equations, Nonlinear Anal., 64 (2006), 21492167.

[3] G. A. Afrouzi, S. Heidarkhani, Existence of three solutions for a class of Dirichlet quasilinear elliptic systems involving the (p1, . . . , pn)-Laplacian, Nonlinear Anal., 70 (2009), 135-143.

[4] L. Boccardo, D. Guedes de Figueiredo, Some remarks on a system of quasilinear elliptic equations, NoDEA Nonlinear Differential Equations Appl., 9 (2002), 309-323.

[5] L. Boccardo, F. Murat Almost everywhere convergence of the gradients of solutions to elliptic and parabolic equations, Nonlinear Anal. 19(6), (1992), 581−597.

[6] O. Claudiandor, L. Alves and M. Marcos. A Lieb type result and application involving a class of non-reflexive Orlicz-Sobolev space.

[7] L. Diening, Theorical and numerical results for electrorheological fluids, Ph.D. thesis, University of Freiburg, Germany (2002)

[8] T. K. Donaldson and N. S. Trudinger, Orlicz-Sobolev Spaces and imbedding theorems. J. Functional Analysis, 8 (1971), 52-75.

[9] N. Fukagai, M. Ito, K. Narukawa, Positive solutions of quasilinear elliptic equations with critical Orlicz-Sobolev nonlinearity on RN, Funkcial. Ekvac., 49 (2006), 235267

[10] A. Kufner, O. John and S. Fucik, Function Spaces, Noordhoff, Leyden, 2013.

[11] M. A. Krasnosel'skii and Ya. B. Rutickii, Convex functions and Orlicz Spaces. Noordhoff. Groningen, 1969.

[12] T. C. Halsey, Electrorheological fluids, Science, 258 (1992), 761-766.

[13] J. Huentutripay, R. Manásevich, Nonlinear eigenvalues for a quasilinear elliptic system in Orlicz-Sobolev Spaces, J. Dynam. Differential Equations, 18 (2006), 901-929.

[14] P. H. Rabinowitz, Minimax methods in critical point theory with applications to differential equations, CBMS Regional Conference Series in Mathematics, Published for the Conference Board of the Mathematical Sciences, Washington, DC; by the American Mathematical Society, Providence, RI, (1986).

[15] L. Wang, X. Zhang, H. Fang, Existence and multiplicity of solutions for a class of quasilinear elliptic systems in Orlicz-Sobolev Spaces. J. Nonlinear Sci. Appl., 10 (2017), 37923814.