Entropy solutions for a nonlinear parabolic problem with lower order terms in Musielak-Orlicz spaces

1st SABIKI HAJAR
Systems Engineering Laboratory
Sultan Moulay Slimane University
National School of Business and Management Beni Mellal, Morocco.
sabikihajar@gmail.com

***Abstract*—We establish an approximation and compactness results in inhomogeneous Musielak-Orlicz-Sobolev spaces, then we shall give the proof of existence results for the entropy solutions of the following nonlinear parabolic problem**

$$\begin{cases} \frac{\partial u}{\partial t} - \mathbf{div}\Big(a(x,t,u,\nabla u)\Big) - \mathbf{div}(\Phi(x,t,u))) = f \quad \text{in } Q_T \\ u(x,0) = u_0(x) \quad \text{in } \Omega \\ u = 0 \quad \text{on } \partial\Omega \times (0,T). \end{cases}$$

Where $Q_T = \Omega \times (0,T)$ and the growth and the coercivity conditions on the monotone vector field a are prescribed by a generalized N-function M. We assume any restriction on M, therefore we work with Musielak-Orlicz spaces which are not necessarily reflexive. The lower order term $\Phi : \Omega \times (0,T) \times \mathbb{R} \to \mathbb{R}^N$ is a Carathéodory function, for a.e. $(x,t) \in Q_T$ and for all $s \in \mathbb{R}$, satisfying only a growth condition and the right hand side f belongs to $L^1(Q_T)$.

***Index Terms*—Non-linear Parabolic problems; Musielak-Orlicz spaces; Entropy Solutions; Non-coercive Problems; Lower order term.**

I. INTRODUCTION

In the last decade, there has been an increasing interest in the study of various mathematical problems in modular spaces. These problems have many consideration in applications (see [14], [38], [41]) and have resulted in a renewal interest in Lebesgue and Sobolev spaces with variable exponent, Musielak, Orlicz space, the origins of which can be traced back to the work of Orlicz in the 1930s. In the 1950s, this study was carried on by Nakano [34] who made the first systematic study of spaces with variable exponent. Later, Polish and Czechoslovak mathematicians investigated the modular function spaces (see for example Musielak [32], Kovacik and Rakosnik [26]). The study of variational problems where the function $a(.)$ satisfies the non-polynomial growth conditions instead of having the usual p-structure arouses much interest with the development of applications to electro-rheological fluids as an important class of non-Newtonian fluids (sometimes referred to as smart fluids). The electro-rheological fluids are characterized by their ability to drastically change the mechanical properties under the influence of an external electromagnetic field. A mathematical model of electro-rheological fluids was proposed by Rajagopal and Ruzicka (we refer to [37], [38] for more details). Another important application is related to image processing [39] where this kind of the diffusion operator is used to underline the borders of the distorted image and to eliminate the noise.

In point of mathematical physics view, it is hard task to show the existence of classical solutions, i.e., solutions which are continuously differentiable as many times as the order of derivatives in equations under consideration. However, the concept of weak solutions is not enough to give a formulation to all problems and does not provide uniqueness and stability properties. Hence, as a certain more general idea, we can use the notion of entropy solution which we have to assume in addition to the weak formulation of the problem certain inequalities.

In this work, we deal with the existence result of the entropy solutions for the following nonlinear parabolic problem without assuming any restriction on the N-function M

$$\begin{cases} \frac{\partial u}{\partial t} - \mathrm{div}\Big(a(x,t,u,\nabla u)\Big) - \mathrm{div}(\Phi(x,t,u)) = f \quad \text{in } Q_T \\ u(x,0) = u_0(x) \quad \text{in } \Omega \\ u = 0 \quad \text{on } \partial\Omega \times (0,T), \end{cases} \tag{1}$$

where the data f belongs to $L^1(Q_T)$, $Au = -\mathrm{div}\Big(a(x,t,u,\nabla u)\Big)$ is a Leray-Lions operator defined on $W_0^{1,x}L_M(Q_T)$. The lower order term $\Phi : \Omega \times (0,T) \times \mathbb{R} \to \mathbb{R}^N$ is a Carathéodory function, for a.e. $(x,t) \in Q_T$ and for all $s \in \mathbb{R}$, satisfying only a growth condition and not necessarily coercive.

The notion of renormalized solution has been introduced by Lions and Di Perna [15] for the study of Boltzmann equation (see also P.-L. Lions [29] for a few applications to fluid mechanics models). This notion was then adapted to elliptic version by Boccardo, J.-L. Diaz, D. Giachetti, F.Murat [13] and F. Murat [31]. At the same the equivalent notion of entropy solutions has been developed independently by Bénilan and al. [11] for the study of nonlinear elliptic problems.

The study of the parabolic equations in Orlicz spaces have been a topic for many years, starting from the work of Donaldson [16] and with later results of Benkirane, Elmahi and Meskine, (see [7], [17], [18]). All of them concern the case of classical spaces, namely Orlicz spaces with an N-function dependent only on x without the dependence on (t,x). We prove our result without any restriction on the growth of an N-function, in particular the Δ_2-condition for an N-function and its conjugate. This results in a need of formulating the approximation theorem and extensively using the notion of modular convergence. The fundamental studies in this direction are due to Gossez for the case of elliptic equations [20], [21]. The appearance of (x,t)-dependence in an N-function requires the studies on the uniform boundedness of the convolution operator. Existence of entropy solution with L^1-data has been proved by Leone and Porretta in [28] for the Dirichlet problem associated to the nonlinear elliptic equation $-\text{div}\Big(a(x,u,\nabla u)\Big) = f$ in the classical Sobolev spaces $W^{1,p}(\Omega)$. In [36] the existence and uniqueness of entropy solutions of the problem (1) has been studied by Prignet where $\Phi = 0$ and $Au = -\text{div}\Big(a(x,t,u,\nabla u)\Big)$ is a Leray-Lions operator in divergence form acting on $W^{1,p}(\Omega)$. The existence of renormalized solutions of the problem (1) in Orlicz spaces has been proved in [24].

As far as we know, there's not much papers concerned with the nonlinear parabolic equations with obstacle in Musielak-Orlicz spaces with L^1 data, in the context of renormalized solution we refer to the work of Gwiazda, Wittbold and al. in [22] where the existence proof related to a nonlinear parabolic problem with L^1-data in Musielak spaces requires a very technical construction of multistage approximation of the solution. In particular it is based on nonlinear semi-group theory of m-accretive operators, but the authors assume that $\overline{M}$ satisfies the Δ_2-condition and the proof was based on the modular Poincaré inequality, we refer also to [23] for the elliptic case without Δ_2-condition on M.

Other difficulties associated with the existence of entropy solutions of the problem (1) lie in the fact that the term $\text{div}(\Phi(x,t,u))$ can not be managed by the divergence theorem and the general Musielak function M does not have to satisfy the suitable condition Δ_2 which induces a loss of reflexivity of our framework setting.

Our main goal of this paper is to prove the existence of an entropy solution of the problem (1) in the sense of Definition 5.1. (see section IV) for a general N-function M.

II. PRELIMENARY

In this section we list briefly some definitions and facts about Musielak-Orlicz-Sobolev spaces. A standard reference is [32]. We also include the definition of inhomogeneous Musielak-Orlicz-Sobolev spaces and some preliminaries lemmas to be used later on this paper.

Musielak-Orlicz spaces: Let Ω be a domain in $\mathbb{R}^d$, $d \in \mathbb{N}$.

Definition 2.1: Let $M\colon \Omega\times\mathbb{R} \mapsto \mathbb{R}$ be a function such that:

(i) For almost all (a. a.) $x \in \Omega$, $M(x,\cdot)$ is an N-function, that is, convex and even in $\mathbb{R}$, increasing in $\mathbb{R}^+$, $M(x,0) = 0$, $M(x,s) > 0$ for all $s > 0$,
$$\lim_{s\to 0}\frac{M(x,s)}{s} = 0, \qquad \lim_{s\to\infty}\frac{M(x,s)}{s} = \infty.$$
(ii) For all $s \in \mathbb{R}$, $M(\cdot,s)$ is a measurable function.

A function $M(x,s)$ which satisfies the conditions (i) and (ii) is called a **Musielak-Orlicz function**, a generalized N-function or a generalized modular function.

From now on, $M\colon \Omega\times\mathbb{R} \mapsto \mathbb{R}$ will stand for a general Musielak-Orlicz function. In some situations, the growth order with respect to t of two given Musielak-Orlicz functions M and P are comparable. This concept is detailed in the next definition.

Definition 2.2: Let $P\colon \Omega\times\mathbb{R} \mapsto \mathbb{R}$ be another Musielak-Orlicz function.

- Assume that there exist two constants $\epsilon > 0$ and $s_0 \geq 0$ such that for a. a. $x \in \Omega$ one has $P(x,s) \leq M(x,\epsilon s)$ for all $s \geq s_0$. Then we write $P \prec M$ and we say that M dominates P globally if $s_0 = 0$ and near infinity if $s_0 > 0$.
- We say that P grows essentially less rapidly than M at $s = 0$ (respectively, near infinity), and we write $P \ll M$, if for every positive constant k_0 we have
$$\limsup_{t\to 0}\sup_{x\in\Omega}\frac{P(x,k_0 s)}{M(x,s)} = 0 \text{ (respectively, } \limsup_{t\to \infty}\sup_{x\in\Omega}\frac{P(x,k_0 s)}{M(x,s)} = 0).$$

We will also use the following notation: $M_x(s) = M(x,s)$, for a. a. $x \in \Omega$ and all $s \in \mathbb{R}$, and we associate its inverse function with respect to $s \geq 0$, denoted by M_x^{-1}, that is,
$$M_x^{-1}(M(x,s)) = M(x, M_x^{-1}(s)) = s, \text{ for all } s \geq 0.$$

Remark 2.3: It is easy to check that $P \ll M$ near infinity if and only if
$$\lim_{s\to\infty}\frac{M_x^{-1}(k_0 s)}{P_x^{-1}(s)} = 0 \text{ uniformly for } x \in \Omega\setminus\Omega_0$$
for some null subset $\Omega_0 \subset \Omega$.□

We introduce the functional $\varrho_{M,\Omega}$ given by
$$\varrho_{M,\Omega}(u) = \int_\Omega M(x,u(x))\,\mathrm{d}x,$$
for any Lebesgue measurable function $u\colon \Omega \mapsto \mathbb{R}$. The set
$$\mathcal{L}_M(\Omega) = \{u\colon \Omega\mapsto\mathbb{R} \text{ mesurable such that } \varrho_{M,\Omega}(u) < \infty\}$$
is called the **Musielak-Orlicz class** related to M in Ω or simply the Musielak-Orlicz class.

The **Musielak-Orlicz space** $L_M(\Omega)$ is the vector space generated by $\mathcal{L}_M(\Omega)$, that is, $L_M(\Omega)$ is the smallest linear space containing the set $\mathcal{L}_M(\Omega)$. Equivalently,
$$L_M(\Omega) = \{u\colon \Omega\mapsto\mathbb{R} \text{ mesurable such that } \varrho_{M,\Omega}(u/\alpha) < \infty, \text{ for some } \alpha > 0\}.$$

For a Musielak-Orlicz function M, we introduce its **complementary function**, denoted by $\bar{M}$, as
$$\bar{M}(x,r) = \sup_{s\geq 0}\{rs - M(x,s)\},$$
that is $\bar{M}(x,r)$ is the complementary to $M(x,s)$ in the sense of Young with respect to the variable r. It turns out that $\bar{M}$ is another Musielak-Orlicz function and the following Young-Fenchel inequality holds
$$|sr| \leq M(x,s) + \bar{M}(x,r) \text{ for all } s,r \in \mathbb{R} \text{ and a. a. } x \in \Omega. \tag{2}$$

In the space $L_M(\Omega)$ we define the following two norms:

$$\|u\|_{M,\Omega} = \inf\left\{\lambda > 0 \,/\, \int_\Omega M(x, u(x)/\lambda)\,\mathrm{d}x \le 1\right\},$$

which is called the **Luxemburg norm**, and the so-called **Orlicz norm**, namely

$$\|u\|_{(M),\Omega} = \sup_{\varrho_{\bar{M},\Omega}(v)\le 1} \int_\Omega u(x)v(x)\,\mathrm{d}x.$$

where the supremum is taken over all $v \in \mathcal{L}_{\bar{M}(\Omega)}$ such that $\varrho_{\bar{M},\Omega}(v) \le 1$. An important inequality in $L_M(\Omega)$ is the following:

$$\int_\Omega M(x, u(x))\,\mathrm{d}x \le \|u\|_{(M),\Omega} \tag{3}$$

for all $u \in L_M(\Omega)$ such that $\|u\|_{(M),\Omega} \le 1$, where from we readily deduce

$$\int_\Omega M\Big(x, \frac{u(x)}{\|u\|_{(M),\Omega}}\Big)\,\mathrm{d}x \le 1 \text{ for all } u \in L_M(\Omega)\setminus\{0\}. \tag{4}$$

From the definition of the Orlicz norm and (2) it is easy to obtain the inequality

$$\|u\|_{(M),\Omega} \le 1 + \int_\Omega M(x, u(x))\,\mathrm{d}x, \text{ for all } u \in L_M(\Omega). \tag{5}$$

It can be shown that the norm $\|\cdot\|_{(M),\Omega}$ is equivalent to the Luxemburg norm $\|\cdot\|_{M,\Omega}$. Indeed,

$$\|u\|_{M,\Omega} \le \|u\|_{(M),\Omega} \le 2\|u\|_{M,\Omega} \text{ for all } u \in L_M(\Omega). \tag{6}$$

Also, Hölder's inequality holds

$$\int_\Omega |u(x)v(x)|\,\mathrm{d}x \le \|u\|_{M,\Omega}\|v\|_{(\bar{M}),\Omega}$$

for all $u \in L_M(\Omega)$ and $v \in L_{\bar{M}}(\Omega)$, Most properties verified by the classical Orlicz spaces cannot be extended to the Musielak-Orlicz spaces unless we assume certain supplementary hypotheses on the generalized N-function M. To this end, we first introduce the two following assumptions.

$$\varrho_{M,\Omega}(\lambda\chi_K) < \infty \text{ for any } \lambda \ge 0 \text{ and any compact set } K \subset \bar{\Omega}. \tag{7}$$

$$\left\{\begin{array}{l} \text{There exist two positive constants } \lambda_0 \text{ and } c_0 \text{ such that} \\ \qquad\qquad \operatorname*{ess\,inf}_{\Omega} M(x, \lambda_0) \ge c_0. \end{array}\right. \tag{8}$$

In (7), χ_A stands for the characteristic function of a measurable set A. The assumption (7) assures that any bounded measurable function with compact support in $\bar{\Omega}$ is in the class $\mathcal{L}_M(\Omega)$. In this situation, we may introduce the space $E_M(\Omega)$ as the closure in $L_M(\Omega)$ of the bounded measurable functions with compact support in $\bar{\Omega}$. The space $E_M(\Omega)$ is then the largest linear space such that $E_M(\Omega) \subset \mathcal{L}_M(\Omega)$, this inclusion being in general strict. Notice that if Ω is bounded then (7) implies the inclusion $L^\infty(\Omega) \subset \mathcal{L}_M(\Omega)$.

On the other hand, the assumption (8) implies that any function in $L_M(\Omega)$ is locally integrable in Ω. This is stated in the following result.

Lemma 2.4: Assume (8). Then the inclusion $L_M(\Omega) \subset L^1_{\mathrm{loc}}(\Omega)$ holds true. Moreover, if $|\Omega| \overset{\mathrm{def}}{=} \mathrm{meas}(\Omega) < \infty$, then $L_M(\Omega) \subset L^1(\Omega)$ with continuous inclusion, that is

$$\|u\|_{L^1(\Omega)} \le C_1\|u\|_{(M),\Omega} \text{ for all } u \in L_M(\Omega),$$

where $C_1 = \lambda_0(|\Omega| + 1/c_0)$.

Proof 2.5: According to the convexity of $M(x,\cdot)$ we obtain

$$sM(x, \lambda_0) \le \lambda_0 M(x, s) \text{ for all } s \ge \lambda_0 \text{ and a. a. } x \in \Omega.$$

Let $u \in L_M(\Omega)$ and $A \subset \Omega$ a measurable set with $|A| < \infty$. Take $\alpha > 0$ such that $\varrho_{M,\Omega}(u/\alpha) < \infty$. Then,

$$\begin{aligned} \int_A \left|\frac{u}{\alpha}\right| &= \int_{A\cap\{|u|<\alpha\lambda_0\}} \left|\frac{u}{\alpha}\right| + \int_{A\cap\{|u|\ge\alpha\lambda_0\}} \left|\frac{u}{\alpha}\right| \\ &\le \lambda_0|A| + \frac{1}{c_0}\int_{A\cap\{|u|\ge\alpha\lambda_0\}} \left|\frac{u}{\alpha}\right| M(x, \lambda_0) \\ &\le \lambda_0|A| + \tfrac{\lambda_0}{c_0}\textstyle\int_\Omega M\Big(x, \frac{u}{\alpha}\Big) < \infty, \end{aligned}$$

and thus $u \in L^1(A)$. If $|\Omega| < \infty$, we may take $A = \Omega$ and $\alpha = \|u\|_{(M),\Omega}$ in the estimate above. Using (4) it yields

$$\int_\Omega |u| \le \lambda_0\Big(|\Omega| + \frac{1}{c_0}\Big)\|u\|_{(M),\Omega}.$$

From now on, we will assume both assumptions (7) and (8) in this paper.

Strong convergence in $L_M(\Omega)$ is rather strict. For most purposes, a mild concept of convergence will be enough, namely, that of **modular convergence**.

Definition 2.6: We say that a sequence $(u_n) \subset L_M(\Omega)$ is modular convergent to $u \in L_M(\Omega)$ if there exists a constant $\lambda > 0$ such that

$$\lim_{n\to\infty} \varrho_{M,\Omega}((u_n - u)/\lambda) = 0.$$

Musielak-Orlicz-Sobolev spaces: According to Lemma 2.4, any function in $L_M(\Omega)$ is locally integrable and, in particular, may be considered as a distribution. This allows us to introduce the so-called Musielak-Orlicz-Sobolev spaces. For any fixed nonnegative integer m we define

$$W^m L_M(\Omega) = \{u \in L_M(\Omega) \,/\, D^\alpha u \in L_M(\Omega) \text{ for all } \alpha, |\alpha| \le m\}$$

where $\alpha = (\alpha_1, \alpha_2, ..., \alpha_d) \in \mathbb{Z}$, $\alpha_j \ge 0$, $j = 1, \dots, d$, with $|\alpha| = \alpha_1 + \alpha_2 + ... + \alpha_d$ and $D^\alpha u$ denote the distributional derivative of multiindex α. The space $W^m L_M(\Omega)$ is called the Musielak-Orlicz-Sobolev space (of order m).

Let $u \in W^m L_M(\Omega)$, we define $\varrho^{(m)}_{M,\Omega}(u) = \sum_{|\alpha|\le m} \varrho_{M,\Omega}(D^\alpha u)$, and

$$\|u\|^{(m)}_{M,\Omega} = \inf\{\lambda > 0 \,/\, \varrho^{(m)}_{M,\Omega}(u/\lambda) \le 1\},$$

$$\|u\|_{m,M,\Omega} = \sum_{|\alpha|\le m} \|D^\alpha u\|_{M,\Omega}.$$

The functional $\varrho^{(m)}_{M,\Omega}$ is convex in $W^mL_M(\Omega)$, whereas the functionals $\|\cdot\|^{(m)}_{M,\Omega}$ and $\|\cdot\|_{m,M,\Omega}$ are equivalent norms on $W^mL_M(\Omega)$. The pair $(W^mL_M(\Omega), \|\cdot\|^{(m)}_{M,\Omega})$ is a Banach space under the assumption (8).

The space $W^mL_M(\Omega)$ is identified to a subspace of the product $\Pi_{|\alpha|\leq m}L_M(\Omega) = \Pi L_M$, this subspace is $\sigma(\Pi L_M, \Pi E_{\bar{M}})$ closed.

Let $W^m_0L_M(\Omega)$ be the $\sigma(\Pi L_M, \Pi E_{\bar{M}})$ closure of $\mathcal{D}(\Omega)$ in $W^mL_M(\Omega)$. Let $W^mE_M(\Omega)$ be the space of functions u such that u and its distribution derivatives up to order m lie in $E_M(\Omega)$, and $W^m_0E_M(\Omega)$ is the (norm) closure of $\mathcal{D}(\Omega)$ in $W^mL_M(\Omega)$.

Since we are going to work with two generalized N-functions, say P and M, such that $P \ll M$, we will consider the following assumptions for both complementary functions $\bar{P}$ and $\bar{M}$:

$$\lim_{|\xi|\to\infty} \operatorname*{ess\,inf}_{x\in\Omega} \frac{\bar{M}(x,\xi)}{|\xi|} = \infty, \tag{9}$$

and

$$\lim_{|\xi|\to\infty} \operatorname*{ess\,inf}_{x\in\Omega} \frac{\bar{P}(x,\xi)}{|\xi|} = \infty. \tag{10}$$

Remark 2.7: From Remark 2.1 in [22] we have that the assumptions (9) and (10) provide the following:

$$\sup_{\xi\in B(0,R)} \operatorname*{ess\,sup}_{x\in\Omega} M(x,\xi) < \infty \text{ for all } 0 < R < +\infty, \tag{11}$$

and

$$\sup_{\xi\in B(0,R)} \operatorname*{ess\,sup}_{x\in\Omega} P(x,\xi) < \infty \text{ for all } 0 < R < +\infty. \tag{12}$$

Also notice that (11) implies (7).

Definition 2.8: We say that a sequence $(u_n) \subset W^1L_M(\Omega)$ converges to $u \in W^1L_M(\Omega)$ for the **modular convergence** in $W^1L_M(\Omega)$ if, for some $h > 0$,

$$\lim_{n\to\infty} \bar{\varrho}^{(1)}_{M,\Omega}((u_n - u)/h) = 0.$$

The following spaces of distributions will also be used:

$$W^{-1}L_{\bar{M}}(\Omega) = \Big\{ f \in \mathcal{D}'(\Omega) \,/\, f = \sum_{|\alpha|\leq 1} (-1)^{|\alpha|} D^\alpha f_\alpha \text{ for some } f_\alpha \in L_{\bar{M}}(\Omega) \Big\}$$

and

$$W^{-1}E_{\bar{M}}(\Omega) = \Big\{ f \in \mathcal{D}'(\Omega) \,/\, f = \sum_{|\alpha|\leq 1} (-1)^{|\alpha|} D^\alpha f_\alpha \text{ for some } f_\alpha \in E_{\bar{M}}(\Omega) \Big\}.$$

Lemma 2.9: If $P \ll M$ and $u_n \to u$ for the modular convergence in $L_M(\Omega)$, then $u_n \to u$ strongly in $E_P(\Omega)$. In particular, $L_M(\Omega) \subset E_P(\Omega)$ and $L_{\bar{P}}(\Omega) \subset E_{\bar{M}}(\Omega)$ with continuous injections.

Proof 2.10: Let $\epsilon > 0$ be given. Let $\lambda > 0$ be such that

$$\int_\Omega M\Big(x, \frac{u_n - u}{\lambda}\Big) \to 0, \text{ as } n \to \infty.$$

Therefore, there exists $h \in L^1(\Omega)$ such that

$$M\Big(x, \frac{u_n - u}{\lambda}\Big) \leq h \text{ and } u_n \to u \text{ a. e. in } \Omega$$

for a sub-sequence still denoted (u_n). Since $P \ll M$, then for all $r > 0$ there exists $t_0 > 0$ such that

$$\frac{P(x, rt)}{M(x,t)} \leq 1, \text{ a. e. in } \Omega \text{ and for all } t \geq t_0.$$

For $r = \frac{\lambda}{\epsilon}$ and $t = \frac{t'}{\lambda}$, we get

$$\frac{P(x, \frac{t'}{\epsilon})}{M(x, \frac{t'}{\lambda})} \leq 1, \text{ when } t' \geq t_0\lambda.$$

Then

$$\begin{aligned} P\Big(x, \frac{u_n - u}{\epsilon}\Big) &\leq M\Big(x, \frac{u_n - u}{\lambda}\Big) + \sup_{t'\in B(0,t_0\lambda)} \operatorname*{ess\,sup}_{x\in\Omega} P(x, t'/\epsilon) \\ &\leq h + \sup_{t'\in B(0,t_0)} \operatorname*{ess\,sup}_{x\in\Omega} P(x, t'/\epsilon) \text{ for a. a. } x \in \Omega. \end{aligned}$$

Since $h + \sup_{t'\in B(0,t_0\lambda)} \operatorname{ess\,sup}_{x\in\Omega} P(x, \frac{t'}{\epsilon}) \in L^1(\Omega)$ (from Remark 2.7), it yields, by the Lebesgue dominated convergence theorem,

$$P\Big(x, \frac{u_n - u}{\epsilon}\Big) \to 0 \text{ in } L^1(\Omega),$$

hence, for n big enough, we have $\|u_n - u\|_{P,\Omega} \leq \epsilon$. That is, $u_n \to u$ in $L_P(\Omega)$.

The continuous injection $L_M(\Omega) \subset E_P(\Omega)$ is trivial since the convergence in $L_M(\Omega)$ implies the modular convergence in this space. On the other hand, since $P \ll M$ is equivalent to $\bar{M} \ll \bar{P}$, this yields the continuous injection $L_{\bar{P}}(\Omega) \subset E_{\bar{M}}(\Omega)$.

Lemma 2.11: (Lemma 2.2 in [30]) Let $(w_n) \subset L_M(\Omega)$, $w \in L_M(\Omega)$, $(v_n) \subset L_{\bar{M}}(\Omega)$ and $v \in L_{\bar{M}}(\Omega)$. If $w_n \to w$ in $L_M(\Omega)$ for the modular convergence and $v_n \to v$ in $L_{\bar{M}}(\Omega)$ for the modular convergence, then

$$\lim_{n\to\infty} \int_\Omega w_n v \, dx = \int_\Omega wv \, dx \text{ and } \lim_{n\to\infty} \int_\Omega w_n v_n \, dx = \int_\Omega wv \, dx.$$

Lemma 2.12: Let $(u_n) \subset E_M(\Omega)$ with $u_n \to u$ in $E_M(\Omega)$. Then there exist $h \in \mathcal{L}_M(\Omega)$ and a subsequence $(u_{n'})$ such that (a. e. stands for 'almost everywhere')

$$|u_{n'}(x)| \leq h(x) \text{ a. e. in } \Omega, \quad \text{and} \quad u_{n'} \to u(x) \text{ a. e. in } \Omega.$$

Proof 2.13: If $u_{n'} = u$ for some subsequence $(u_{n'})$, then the result is trivial. Thus, we may assume that for $n \geq 1$ large enough (and some subsequence, if necessary, still denoted in the same way) it is $0 < 2\|u_n - u\|_{(M)} \leq 1$. Then

$$\begin{aligned} \|M_x(2(u_n - u)\|_{L^1(\Omega)} &= \int_\Omega M_x\Big(2\|u_n - u\|_{(M)} \frac{u_n - u}{\|u_n - u\|_{(M)}}\Big) \\ &\leq 2\|u_n - u\|_{(M)} \int_\Omega M_x\Big(\frac{u_n - u}{\|u_n - u\|_{(M)}}\Big) \\ &\leq 2\|u_n - u\|_{(M)}. \end{aligned}$$

Thus, $\|M_x(2(u_n - u)\|_{L^1(\Omega)} \to 0$, as $n \to \infty$. Therefore there exists $h_1 \in L^1(\Omega)$ and a subsequence $(u_{n'})$ such that

$u_{n'} \to u(x)$ a. e. in Ω and $M_x(2(u_{n'}(x) - u(x)) \le h_1(x)$ a. e. in Ω, which implies that

$$|u_{n'}| \le |u(x)| + \frac{1}{2} M_x^{-1}(h_1(x)) \overset{\text{def}}{=} h(x).$$

Since

$$\int_\Omega M_x\Big(|u(x)| + \frac{1}{2} M_x^{-1}(h_1(x))\Big) \le \frac{1}{2}\int_\Omega M_x(2u(x)) + \frac{1}{2}\int_\Omega h_1(x) < \infty,$$

we finally obtain $h \in \mathcal{L}_M(\Omega)$.

Lemma 2.14: (Cf. [4]) Let Ω be a bounded and Lipschitz-continuous domain in $\mathbb{R}^d$ and let M and $\bar{M}$ be two complementary Musielak-Orlicz functions in $\Omega \times \mathbb{R}$ which satisfy the following conditions:

(i) There exists a constant $A > 0$ such that for all $x, y \in \Omega$ with $0 < |x - y| \le \frac{1}{2}$ one has

$$\frac{M(x,s)}{M(y,s)} \le s^{-\frac{A}{\log|x-y|}} \text{ for all } s \ge 1. \tag{13}$$

(ii) There exists a constant $C > 0$ such that

$$\bar{M}(x,1) \le C \text{ a. e. in } \Omega. \tag{14}$$

Then the space $\mathcal{D}(\Omega)$ is dense in $L_M(\Omega)$ with respect to the modular convergence, $\mathcal{D}(\Omega)$ is dense in $W_0^1 L_M(\Omega)$ for the modular convergence and $\mathcal{D}(\bar{\Omega})$ is dense in $W^1 L_M(\Omega)$ for the modular convergence.

Remark 2.15: By taking $s = 1$ in (13) it yields that $M(x,1) =$ constant for a. a. $x \in \Omega$. In particular, the condition (8) is obviously verified and also

$$\int_\Omega M(x,1)\,\mathrm{d}x < \infty.$$

Remark 2.16: (Cf. [9]) Let $p\colon \Omega \mapsto (1,\infty)$ be a measurable function such that there exists a constant $A > 0$ such that for all points $x, y \in \Omega$ with $|x - y| < 1/2$, one has the inequality

$$|p(x) - p(y)| \le -\frac{A}{\log|x-y|}.$$

Then the following Musielak-Orlicz functions satisfy the assumption (13):

1) $M(x,s) = s^{p(x)}$;
2) $M(x,s) = s^{p(x)}\log(1+s)$;
3) $M(x,s) = s\log(1+s)(\log(e-1+s))^{p(x)}$.

Poincaré's inequality does not hold in generalized Orlicz-Sobolev spaces unless the Musielak-Orlicz function $M(x,s)$ verifies some structural assumption. To this end, we introduce the following definition [3].

Definition 2.17: A generalized function $M(x,s)$ is said to satisfy the Y-condition on a non-empty bounded interval $(a,b) \subset \mathbb{R}$, if either

(Y_0) there exist $s_0 \ge 0$ and $1 \le i \le N$ such that the function $x_i \in (a,b) \mapsto M(x,s)$ changes constantly its monotony on both sides of s_0 (that is, for $s \ge s_0$ and $0 \le s < s_0$),

or

(Y_∞) there exists $1 \le i \le N$ such that for all $s \ge 0$, the function $x_i \in (a,b) \mapsto M(x,s)$ is monotone on (a,b).

Here, x_i stands for the i-th component of $x \in \Omega$.

Lemma 2.18: (Poincaré's inequality [3]) Let Ω be a bounded and Lipschitz-continuous domain in $\mathbb{R}^d$ and let M and $\bar{M}$ be two complementary Musielak-Orlicz functions in $\Omega \times \mathbb{R}$. Assume that M verifies (13) and the Y-condition, and also that $\bar{M}$ verifies (7) and (14). Then there exists a constant $C_0 = C_0(\Omega, M) > 0$ such that

$$\|u\|_{M,\Omega} \le C_0 \|\nabla u\|_{M,\Omega}, \text{ for all } u \in W_0^1 L_M(\Omega). \tag{15}$$

From this point on we will always assume that the hypothesis of Lemma 2.18 hold true.

Remark 2.19: Let M be a Musielak-Orlicz function such that (15) is verified and let $u \in W_0^1 L_M(\Omega)$. Assume that, for some constant $C \ge 0$, one has $\int_\Omega M(x, \nabla u)\,\mathrm{d}x \le C$. Then, $\|u\|_{1,M,\Omega} \le C'$ where $C' = (C_0 + 1)\max(C, 1)$. Indeed, since $\|u\|_{1,M,\Omega} = \|u\|_{M,\Omega} + \|\nabla u\|_{M,\Omega}$, by using (15), we get

$$\|u\|_{1,M,\Omega} \le C_0\|\nabla u\|_{M,\Omega} + \|\nabla u\|_{M,\Omega} \le (C_0 + 1)\|\nabla u\|_{M,\Omega}.$$

Now, if $C \ge 1$, according to the convexity of $M(x,\cdot)$, it yields

$$\int_\Omega M\Big(x, \frac{\nabla u}{C}\Big)\,\mathrm{d}x \le \frac{1}{C}\int_\Omega M(x,\nabla u)\,\mathrm{d}x \le \frac{C}{C} = 1,$$

this means that $C \in \{\lambda > 0, \int_\Omega M(x, \nabla u/\lambda)\,\mathrm{d}x \le 1\}$, hence $\|\nabla u\|_{M,\Omega} \le C$. On the other hand, if $C < 1$, then $\int_\Omega M(x,\nabla u)\,\mathrm{d}x \le C < 1$, which yields $\|\nabla u\|_{M,\Omega} \le 1$.

Inhomogeneous Musielak-Orlicz-Sobolev spaces. When dealing with parabolic equations in the context of Musielak-Orlicz-Sobolev spaces we need to introduce some particular spaces which take into account the different orders of differentiation with respect to the spatial variables and the time variable.

Let Ω be a bounded and open subset of $\mathbb{R}^d$ and let $Q_T = \Omega \times (0,T)$ for some $T > 0$. Let $M = M(x,s)$ be a Musielak-Orlicz function in $\Omega \times \mathbb{R}$ (here we do not consider a more general case where $M = M(x,t,s)$, $(x,t) \in Q_T$). For each $\alpha = (\alpha_1, \dots, \alpha_d) \in \mathbb{Z}^d$, $\alpha_j \ge 0$, $j = 1, \dots, d$, we denote by D_x^α the distributional derivative on Q_T of multiindex α with respect to the variable $x \in \mathbb{R}^d$. The inhomogeneous Musielak-Orlicz-Sobolev spaces of order one are defined as follows:

$$W^{1,x}L_M(Q_T) = \{u \in L_M(Q_T) \,/\, D_x^\alpha u \in L_M(Q_T) \text{ for all } \alpha,\ |\alpha| \le 1\}$$

and

$$W^{1,x}E_M(Q_T) = \{u \in E_M(Q_T) \,/\, D_x^\alpha u \in E_M(Q_T) \text{ for all } \alpha,\ |\alpha| \le 1\}$$

This last space is a subspace of the first one, and both are Banach spaces under the assumption (8) and with norm

$$\|u\| = \sum_{|\alpha| \le 1} \|D_x^\alpha u\|_{M,Q_T}.$$

These spaces are considered as subspaces of the product space $\Pi L_M(Q_T)$ which has $(d+1)$ copies.

We shall also consider the weak-$*$ topologies $\sigma(\Pi L_M(Q_T), \Pi E_{\overline{M}}(Q_T))$ and $\sigma(\Pi L_M(Q_T), \Pi L_{\overline{M}}(Q_T))$. If $u \in W^{1,x}L_M(Q_T)$ then the function $t :\rightarrow u(t)$ is defined on $(0,T)$ with values in $W^1L_M(\Omega)$. If, further, $u \in W^{1,x}E_M(Q_T)$ then this function is a $W^1E_M(\Omega)$-valued and is strongly measurable. The space $W^{1,x}L_M(Q_T)$ is not in general separable. If $u \in W^{1,x}L_M(Q_T)$, we cannot conclude that the function $u(t)$ is measurable on $(0,T)$. However, the scalar function $t \rightarrow \|u(t)\|_{M,\Omega}$ is in $L^1(0,T)$. The space $W_0^{1,x}E_M(Q_T)$ is defined as the (norm) closure in $W^{1,x}E_M(Q_T)$ of $\mathcal{D}(Q)$. We can easily show as in [4] that when Ω is a Lipschitz-continuous domain then each element u of the closure of $\mathcal{D}(Q_T)$ with respect of the weak-$*$ topology $\sigma(\Pi L_M, \Pi E_{\bar{M}})$ is limit, in $W^{1,x}L_M(Q_T)$, of some subsequence $(u_n) \subset \mathcal{D}(Q_T)$ for the modular convergence; i. e., there exists $\lambda > 0$ such that for all α with $|\alpha| \leq 1$

$$\int_{Q_T} M\Big(x, \frac{D_x^\alpha u_n - D_x^\alpha u}{\lambda}\Big)\, \mathrm{d}x\, \mathrm{d}t \rightarrow 0 \ as \ n \ \rightarrow \infty,$$

and, in particular, this implies that (u_n) converges to u in $W^{1,x}L_M(Q_T)$ for the weak-$*$ topology $\sigma(\Pi L_M, \Pi L_{\bar{M}})$. Consequently

$$\overline{\mathcal{D}(Q_T)}^{\sigma(\Pi L_M, \Pi L_{\bar{M}})} = \overline{\mathcal{D}(Q_T)}^{\sigma(\Pi L_M, \Pi E_{\bar{M}})}.$$

This space will be denoted by $W_0^{1,x}L_M(Q_T)$. Furthermore,

$$W_0^{1,x}E_M(Q_T) = W_0^{1,x}L_M(Q_T) \cap \Pi E_{\bar{M}}(Q_T).$$

Poincaré's inequality also holds in $W_0^{1,x}L_M(Q_T)$, i. e. there exists a constant $C > 0$ such that for all $u \in W_0^{1,x}L_M(Q_T)$ one has

$$\sum_{|\alpha|\leq 1} \|D_x^\alpha u\|_{M,Q_T} \leq C \sum_{|\alpha|=1} \|D_x^\alpha u\|_{M,Q_T}. \tag{16}$$

The dual space of $W_0^{1,x}E_M(Q_T)$ will be denoted by $W^{-1,x}L_{\bar{M}}(Q_T)$, and it can be shown that

$$W^{-1,x}L_{\bar{M}}(Q_T) = \left\{ f = \sum_{|\alpha|\leq 1} D_x^\alpha f_\alpha \,/\, f_\alpha \in L_{\bar{M}}(Q_T), \right\}.$$

for all α.
This space will be equipped with the usual quotient norm

$$\|f\| = \inf \sum_{|\alpha|\leq 1} \|D_x^\alpha f_\alpha\|_{\bar{M},Q_T}$$

where the infimum is taken over all possible functions $f_\alpha \in L_{\bar{M}}(Q_T)$ from which the decomposition $f = \sum_{|\alpha|\leq 1} D_x^\alpha f_\alpha$ holds true.

We also denote by $W^{-1,x}E_{\bar{M}}(Q_T)$ the subspace of $W^{-1,x}L_{\bar{M}}(Q_T)$ consisting of those linear forms which are $\sigma(\Pi L_M, \Pi E_{\bar{M}})$-continuous. It can be shown that

$$W^{-1,x}E_{\bar{M}}(Q_T) = \left\{ f = \sum_{|\alpha|\leq 1} D_x^\alpha f_\alpha \,/\, f_\alpha \in E_{\bar{M}}(Q_T) \right\}.$$

III. Compactness results

In the sequel, we will make use of the following results which concern mollification with respect to time and space variables and some trace results. For a function $u \in L^1(Q_T)$ we introduce the function $\tilde{u} \in L^1(\Omega \times \mathbb{R})$ as $\tilde{u}(x,s) = u(x,s)\chi_{(0,T)}$ and define, for all $\mu > 0$, $t \in [0,T]$ and a.e. $x \in \Omega$, the function u_μ given as follows

$$u_\mu(x,t) = \mu \int_{-\infty}^{t} \tilde{u}(x,s)\exp(\mu(s-t))\, \mathrm{d}s. \tag{17}$$

Lemma 3.1: ([2]).

1) Let $u \in L_M(Q_T)$. Then $u_\mu \in C([0,T]; L_M(\Omega))$ and $u_\mu \rightarrow u$ as $\mu \rightarrow +\infty$ in $L_M(Q_T)$ for the modular convergence.
2) Let $u \in W^{1,x}L_M(Q_T)$. Then $u_\mu \in C([0,T]; W^1L_M(\Omega))$ and $u_\mu \rightarrow u$ as $\mu \rightarrow +\infty$ in $W^{1,x}L_M(Q_T)$ for the modular convergence.
3) Let $u \in E_M(Q_T)$ (respectively, $u \in W^{1,x}E_M(Q_T)$). Then $u_\mu \rightarrow u$ as $\mu \rightarrow +\infty$ strongly in $E_M(Q_T)$ (respectively, strongly in $W^{1,x}E_M(Q_T)$).
4) Let $u \in W^{1,x}L_M(Q_T)$ then $\frac{\partial u_\mu}{\partial t} = \mu(u - u_\mu) \in W^{1,x}L_M(Q_T)$.
5) Let $(u_n) \subset W^{1,x}L_M(Q_T)$ and $u \in W^{1,x}L_M(Q_T)$ such that $u_n \rightarrow u$ strongly in $W^{1,x}L_M(Q_T)$ (respectively, for the modular convergence). Then, for all $\mu > 0$, $(u_n)_\mu \rightarrow u_\mu$ strongly in $W^{1,x}L_M(Q_T)$ (respectively, for the modular convergence).

Lemma 3.2: [2] Let M be a Musielak function. Let Y be a Banach space such that the following continuous imbedding holds $L^1(\Omega) \subset Y$. Then, for all $\epsilon > 0$ and all $\lambda > 0$ there is C_ϵ such that for all $u \in W^{1,x}L_M(Q_T)$ with $\frac{\nabla u}{\lambda} \in K_M(Q_T)$

$$\|u\|_{L^1(\Omega)} \leq \epsilon\lambda\Big(\int_{Q_T} M(x, \frac{\nabla u}{\lambda})\, dx\, dt + T\Big) + C_\epsilon \|u\|_{L^1(0,T;Y)}. \tag{18}$$

Lemma 3.3: [2] Let Y be a Banach space such that $L^1(\Omega) \subset Y$ with continuous imbedding.
If F is bounded in $W_0^{1,x}L_M(Q_T)$ and is relatively compact in $L^1(0,T;Y)$ then F is relatively compact in $L^1(Q_T)$.

Lemma 3.4: (cf. [33]) Let $Q_T = \Omega \times (0,T)$, let M a Musielak-Orlicz function, $E_M(\Omega)$ the Musielak-Orliccz space on Ω and $E_M(Q_T)$ the inhomogeneous Musielak-Orlicz space on Q_T. Then there embeddings map

$$E_M(Q_T) \subseteq L^1(0,T; E_M(\Omega)). \tag{19}$$

Lemma 3.5: Let $Q_T = \Omega \times (0,T)$, let M a Musielak-Orlicz function, $W^1E_M(\Omega)$ the Musielak-Orliccz-Sobolev space on Ω and $W^1E_M(Q_T)$ the inhomogeneous Musielak-Orlicz-Sobolev space on Q_T. Then the following embeddings

$$W^1E_M(Q_T) \subset L^1\left(0,T; W^1E_M(\Omega)\right) \tag{20}$$

$$W^{-1}E_{\overline{M}}(Q_T) \subset L^1\left((0,T); W^{-1}E_{\overline{M}}(\Omega)\right) \tag{21}$$

are continuous

Proof 3.6: Let $u \in W^1E_M(Q_T))$, we have $u \in E_M(Q_T)$ and $D_x^\alpha u \in E_M(Q_T)$. By the previous lemma, we get

$$\int_0^T ||u||_{M,\Omega}\, dt \leq (T+1)||u||_{M,Q_T}, \tag{22}$$

and

$$\int_0^T ||D_x^\alpha u||_{M,\Omega}\, dt \leq (T+1)||D_x^\alpha u||_{M,Q_T} \quad \text{for all } |\alpha| \leq 1, \tag{23}$$

which implies

$$\int_0^T ||u||_{L^1(0,T;W^1E_M(\Omega))}\, dt \leq (T+1)||u||_{W^{1,x}E(Q_T)}. \tag{24}$$

Consequently (20) is proved.
Using the same Technics we will prove (21). Since every $f \in W^{-1,x}E_{\overline{M}}(Q_T)$ reads as

$$f = \sum_{|\alpha|\leq 1} D_x^\alpha g_\alpha \text{ where } g_\alpha \in E_{\overline{M}}(Q_T)$$

and

$$\|f\|_{W^{-1,x}L_{\overline{M}}(Q_T)} = \sum_{|\alpha|\leq 1} \|g_\alpha\|_{\overline{M},Q_T}.$$

This gives

$$\int_0^T \sum_{|\alpha|\leq 1} \|g_\alpha(t)\|_{\overline{M},\Omega} \leq (1+T)\|f\|_{W^{-1,x}L_{\overline{M}}(Q_T)},$$

by definition of the quotient norm of $W^{-1}L_{\overline{M}}(\Omega)$ we have

$$\|f(t)\|_{W^{-1}L_{\overline{M}}(\Omega)} \leq \sum_{|\alpha|\leq 1} \|g_\alpha(t)\|_{\overline{M},\Omega},$$

and then

$$\int_0^T \|f(t)\|_{W^{-1}L_{\overline{M}}(\Omega)} dt \leq (T+1)\|f\|_{W^{-1,x}L_{\overline{M}}(Q_T)}.$$

This gives the desired result.

Theorem 3.7: [2] Let M be a Musielak function. If F is bounded in $W_0^{1,x}L_M(Q_T)$ and $\frac{\partial f}{\partial t} : f \in F$ is bounded in $W^{-1,x}L_{\overline{M}}(Q_T)$, then F is relatively compact in $L1(Q)$.

Lemma 3.8: [40] Let B be a Banach space.
If $f \in \mathcal{D}'(]0,T[;B)$ is such that $\frac{\partial f}{\partial t} \in L^1(0,T;B)$ then $f \in C(]0,T[;B)$ and for all $h > 0$ we have $||\tau_h(f) - f||_{L^1(0,T;B)} \leq h||\frac{\partial f}{\partial t}||_{L^1(0,T;B)}$.

Remark 3.9: By the Theorem 3.4, if $F \subset L^1(0,T;B)$ is such that $\left\{\frac{\partial f}{\partial t} : \quad f \in F\right\}$ is bounded in $L^1(0,T;B)$ then $||\tau_h(f) - f||_{L^1(0,T;B)} \to 0$ as $h \to 0$ uniformly with respect to $f \in F$.

Corollary 3.10: Let M be a Musielak-Orlicz function. Let (u_n) be a sequence of $W^{1,x}L_M(Q_T)$ such that

$$u_n \rightharpoonup u \quad \text{weakly in } W^{1,x}L_M(Q_T) \quad \text{for} \quad \sigma(\Pi L_M, \Pi E_{\overline{M}})$$

and

$$\frac{\partial u_n}{\partial t} = h_n + k_n \quad \text{in } \mathcal{D}'(Q_T)$$

with (h_n) bounded in $W^{-1,x}L_{\overline{M}}(Q_T)$ and (k_n) bounded in the space $L^1(Q_T)$ of measures on Q_T. Then

$$u_n \to u \quad \text{strongly in} \quad L^1_{loc}(Q_T).$$

If further $u_n \in W_0^{1,x}L_M(Q_T)$ then $u_n \to u$ in $L^1(Q_T)$.

Proof 3.11: The proof is easily adapted from that given in [12] by using Theorem 3.7 and Remark 3.9 instead of lemma [40].

IV. Existence result

Let Ω be a bounded Lipschitz domain in $\mathbb{R}^N$ $(N \geq 2)$, $T > 0$ and set $Q_T = \Omega \times [0,T]$. We denote $Q_\tau = \Omega \times [0,\tau]$. Let M and P two Musielak-Orlicz functions such that $P \ll M$ and their conjugate respectively $\overline{M}$ and $\overline{P}$ satisfy (9) and (10).
Consider a second-order partial differential operator

$$A: \; D(A) \subset W^{1,x}L_M(Q_T) \to \; W^{-1,x}L_{\overline{M}}(Q_T)$$

in divergence form

$$A(u) = -\text{div}(a(x,t,u,\nabla u))$$

where

$$a : \Omega\times\mathbb{R}\times\mathbb{R}^N \to \mathbb{R}^N \text{is a Carathéodory function satisfying} \tag{25}$$

for almost every $(x,t) \in Q_T$ and all $s \in \mathbb{R},\ \xi \neq \xi \in \mathbb{R}^N$

$$|a(x,t,s,\xi)| \leq \beta(c_1(x,t)+\overline{M}_x^{-1}P(x,k_1|s|+\overline{M}_x^{-1}M(x,k_1|\xi|) \tag{26}$$

$$[a(x,t,s,\ \xi) - a(x,t,s,\ \xi')][\xi - \xi'] \ > 0 \tag{27}$$

$$a(x,t,s,\ \xi)\xi \geq \alpha[M(x,|s|) + M(x,|\xi|)] \tag{28}$$

with $c_1(x,t) \in E_{\overline{M}}(Q_T)$, $c(x,t) \geq 0$ and $\alpha, \beta, k > 0$.
The function ϕ is a Carathéodory function satisfing the following conditions

$$|\Phi(x,t,s)| \leq \gamma(x,t)\overline{P}_x^{-1}P(x,|s|), \tag{29}$$

with $\gamma \in L^\infty(Q_T)$

$$f \in L^1(Q_T) \tag{30}$$

$$u_0 \in L^1(\Omega). \tag{31}$$

Lemma 4.1: Under assumptions (25)- (28), let (z_n) be a sequence in $W_0^{1,x}L_M(Q_T)$ such that,

(i) $z_n \rightharpoonup z$ in $W_0^{1,x}L_M(Q_T)$ for $\sigma(\Pi L_M(Q_T), \Pi E_{\overline{M}})$
(ii) $(a(x,t,z_n,\nabla z_n))_n$ is bounded in $(L_M(Q_T))^N$
(iii) $\displaystyle\int_{Q_T} [a(x,t,z_n,\nabla z_n) - a(x,t,z_n,\nabla z\chi_s)][\nabla z_n - \nabla z\chi_s]\, dx\, dt \to 0.$ (32)

as n and s tend to ∞, and where χ is the characteristic function of

$$Q_s = \{(x,t) \in Q_T; |\nabla z| \leq s\}$$

Then,

$$\nabla z_n \to \nabla z \quad \text{a.e. in} \quad Q_T, \tag{33}$$

$$\lim_{n\to\infty}\int_{Q_T}[a(x,t,z_n,\nabla z_n)\nabla z_n\,dx\,dt=\int_{Q_T}[a(x,t,z,\nabla z)\nabla z\,dx\,dt \quad (34)$$

$$M(x,|\nabla z_n|)\to M(x,|\nabla z|) \text{ strongly in } L^1(Q_T) \quad (35)$$

Proof 4.2: We proceed as in the case of Orlicz spaces (see [1]), we get the desired result.

V. Definition of an entropy solution.

The definition of an entropy solution for problem (1) can be stated as follows.

Definition 5.1:

A measurable function $u:\Omega\times(0,T)\to\mathbb{R}$ is called entropy solution of (1) if u belongs to $L^\infty(0,T;L^1(\Omega))$, $T_K(u)$ belongs to $D(A)\cap W_0^{1,x}L_M(Q_T)$ for every $K>0$, $\Theta_K(u(.,t))$ belongs to $L^1(\Omega)$ for every $t\in[0,T]$ and for every $K>0$ and u satisfies :

$$\begin{aligned}&\int_\Omega \Theta_K(u-v)dx+\langle\frac{\partial v}{\partial t},T_K(u-v)\rangle_{Q_\tau}\\&+\int_{Q_\tau}a(x,t,T_K(u),\nabla T_K(u))\nabla T_K(u-v)\,dx\,dt\\&+\int_{Q_\tau}\Phi(x,t,u)\nabla T_K(u-v)\,dx\,dt\\&\le\int_{Q_\tau}fT_K(u-v)\,dx\,dt+\int_\Omega\Theta_K(u_0-v(0))dx,\end{aligned} \quad (36)$$

and

$$u(x,0)=u_0(x) \quad \text{for a.e } x\in\Omega, \quad (37)$$

for every $\tau\in[0,T]$, $K>0$ and for all $v\in W_0^{1,x}L_M(Q_T)\cap L^\infty(Q_T)$ such that $\frac{\partial v}{\partial t}$ belongs to $W^{-1,x}L_{\overline{M}}(Q_T)+L^1(Q_T)$ (recall that $\Theta_K(r)=\int_0^r T_K(r)dr$ is the primitive of the usual truncation T_K).

This section is devoted to establish the following existence theorem

Theorem 5.2: Assume that the hypotheses (25)-(29) are satisfied, then there exists at least one solution of problem (1) in the sens of Definition (36).

Proof 5.3:

Step1 : Approximation problem.

Let f_n and u_{0n} regular functions in $L^1(Q_T)$ (resp $L^1(\Omega)$) such that:

$$f_n\longrightarrow f \text{ in } L^1(Q_T) \text{ and } \|f_n\|_{L^1}\le\|f\|_{L^1} \quad (38)$$

and

$$\|u_{0n}\|_{L^1}\le\|u_0\|_{L^1} \text{ and } u_{0n}\longrightarrow u_0 \text{ in } L^1(\Omega), \quad (39)$$

as n tends to $+\infty$.

Now, we consider the following regularized problem

$$\begin{cases}\frac{\partial u_n}{\partial t}-\operatorname{div}\Big(a(x,t,u_n,\nabla u_n)\Big)-\operatorname{div}(\Phi(x,t,u_n))=f_n \text{in } Q_T\\ u_n(x,0)=u_{0n}(x) \quad \text{in } \Omega\\ u_n=0 \quad \text{on } \partial\Omega\times(0,T),\end{cases} \quad (40)$$

The problem (40) can be written as follows

$$\begin{cases}\frac{\partial u_n}{\partial t}-\operatorname{div}\Big(F_n(x,t,u_n,\nabla u_n)\Big)=f_n \quad \text{in } Q_T\\ u_n(x,0)=u_{0n}(x) \quad \text{in } \Omega\\ u_n=0 \quad \text{on } \partial\Omega\times(0,T),\end{cases} \quad (41)$$

with $F_n(x,t,u_n,\nabla u_n)=a(x,t,u_n,\nabla u_n)+(\Phi(x,t,u_n)$.
Note that F_n satisfies the assumptions (A_1), (A_2) and (A_3) as in [27].
Indeed, using (26), (27)and (29) we deduce that F_n satisfies (A_1), (A_2), it remains to prove (A_3). Let $u_n\in W_0^{1,x}L_M(Q_T)$ by (29) and Young inequality we obtain

$$\begin{aligned}|\Phi(x,t,u_n)\nabla u_n|&\le|\gamma(x,t)|(P(x,|u_n|)+P(x,|\nabla u_n|))\\&\le C_\gamma(P(x,|u_n|)+P(x,|\nabla u_n|)).\end{aligned} \quad (42)$$

$P\ll M$, then we have for all $\varepsilon>0$ there exists t_0 that

$$P(x,t)\le M(x,\varepsilon t) \text{ for all } t\ge t_0,\ a.e. x\in\Omega. \quad (43)$$

Let

$$E_1=\{(x,t)\in Q_T;|u_n(x,t)|\ge t_0\}$$

$$\text{and } E_2=\{(x,t)\in Q_T;|\nabla u_n(x,t)|\ge t_0\}$$

Case 1 : if $(x,t)\in E_1\cap E_2$
In virtue of (42) and (43), we have

$$|\Phi(x,t,u_n)\nabla u_n|\le C_\gamma(M(x,\varepsilon|u_n|)+M(x,\varepsilon|\nabla u_n|)). \quad (44)$$

Without loss of generality, we can assume that $\varepsilon=\frac{\alpha}{2C_\gamma+\alpha}$ which is $\varepsilon\le1$, then by convexity of the function $M(x,.)$, one has

$$\begin{aligned}|\Phi(x,t,u_n)\nabla u_n|&\le C_\gamma\varepsilon(M(x,|u_n|)+M(x,|\nabla u_n|))\\&\le\frac{\alpha}{2}(M(x,|u_n|)+M(x,|\nabla u_n|)),\end{aligned} \quad (45)$$

which implies

$$\Phi(x,t,u_n)\nabla u_n\ge-\frac{\alpha}{2}(M(x,|u_n|)+M(x,|\nabla u_n|)). \quad (46)$$

From (28) and (46), we have

$$F_n(x,t,u_n,\nabla u_n).\nabla u_n\ge\frac{\alpha}{2}M(x,|\nabla u_n|). \quad (47)$$

Case 2 : if $(x,t)\in E_1^c\cap E_2^c$
We have

$$|\Phi(x,t,u_n)\nabla u_n|\le C_\gamma(P(x,|u_n|)+P(x,|\nabla u_n|)) \quad (48)$$

Using the Remark 2.7, we obtain

$$P(x,|u_n|)\le ess\sup_{x\in\Omega}P(x,t_0)<R_1<\infty \quad (49)$$

and

$$P(x,|\nabla u_n|)\le ess\sup_{x\in\Omega}P(x,t_0)<R_2<\infty. \quad (50)$$

From (49) and (50) we get

$$|\Phi(x,t,u_n)\nabla u_n| \leq C_0. \tag{51}$$

By (28) and (51) we deduce

$$F_n(x,t,u_n,\nabla u_n).\nabla u_n \geq \alpha M(x,|\nabla u_n|) - C_0 \tag{52}$$

Case 3 :if $(x,t) \in E_1^c \cap E_2$.
In this case, by using Remark 2.7 and (43) we get :

$$|\Phi(x,t,u_n)\nabla u_n| \leq C_1 + C_\gamma M(x, r|\nabla u_n|). \tag{53}$$

We can assume again that $r = \frac{\alpha}{2C_\gamma + \alpha}$ which is $r \leq 1$, then by convexity of the function $M(x,.)$, one has

$$\Phi(x,t,u_n)\nabla u_n \geq -\frac{\alpha}{2} M(x,|\nabla u_n|) - C_1.$$

which implies by using (27)

$$\begin{aligned} F_n(x,t,u_n,\nabla u_n).\nabla u_n &\geq \frac{\alpha}{2} M(x,|\nabla u_n|) + \alpha M(x,|u_n|) - C_1 \\ &\geq \frac{\alpha}{2} M(x,|\nabla u_n|) - C_1. \end{aligned} \tag{54}$$

By the same way if $(x,t) \in E_1 \cap E_2^c$ we get

$$\begin{aligned} F_n(x,t,u_n,\nabla u_n).\nabla u_n &\geq \frac{\alpha}{2} M(x,|u_n|) + \alpha M(x,|\nabla u_n|) - C_2. \\ &\geq \alpha M(x,|\nabla u_n|) - C_2. \end{aligned} \tag{55}$$

Finally, from (47), (52) and (54) the assumption (A_3) in [27] is true.
Then there exists at least one solution u_n of (40), (the existence of u_n can be obtained from Galerkin solutions corresponding to the equation (40) as in [27], see Theorem 1 of [2] for more details).

Step 2 : A priori estimates.

Lemma 5.4: Suppose that the assumptions (25) - (29) are true and let u_n be a solution of the approximate problem (40). Then for all $K, n > 0$, we have

$$\int_{Q_T} M(x, |\nabla T_K(u_n)|)\,dx\,dt \leq CK. \tag{56}$$

Where C is a positive constant independent of n and K.
And

$$\lim_{K\to\infty} mes\,\{(x,t) \in Q_T; |u_n| > K\} = 0. \tag{57}$$

Proof 5.5: Let us note that in the following of this work we will set

$$\Theta_K(t) = \int_0^t T_K(s)ds \tag{58}$$

the primitive of the truncated function $T_K(s)$.
Taking $v = T_K(u_n)_{\chi(0,\tau)}$ as test function in the equation (40) we obtain

$$\begin{aligned} &\int_\Omega \Theta_K(u_n)(\tau)dx - \int_\Omega \Theta_K(u_{0n})dx \\ &+ \int_{Q_\tau} a(x,t,u_n,\nabla u_n)\nabla T_K(u_n)\,dx\,dt \\ &+ \int_{Q_\tau} \Phi(x,t,u_n)\nabla T_K(u_n)\,dx\,dt \\ &= \int_{Q_\tau} f_n T_K(u_n)\,dx\,dt, \end{aligned} \tag{59}$$

since $\nabla T_K(u_n) = 0$ in set $\{(x,t) \in Q_T;\ \ |u_n(x,t)| > K\}$ which implies that

$$\begin{aligned} &\int_\Omega \Theta_K(u_n)(\tau)dx + \int_{Q_\tau} a(x,t,T_K(u_n),\nabla T_K(u_n))\nabla T_K(u_n)\,dx\,dt \\ &\quad + \int_{Q_\tau} \Phi(x,t,T_K(u_n))\nabla T_K(u_n)\,dx\,dt \\ &= \int_{Q_\tau} f_n T_K(u_n)\,dx\,dt + \int_\Omega \Theta_K(u_{0n})dx. \end{aligned} \tag{60}$$

First, from (38) and (39) we have

$$\int_{Q_\tau} f_n T_K(u_n)\,dx\,dt + \int_\Omega \Theta_K(u_{0n})dx \leq K(||f||_{1,Q_T} + ||u_0||_{1,\Omega}) \equiv C_2 K, \tag{61}$$

where $C_2 = (||f||_{L^1}(Q_T) + ||u_0||_{L^1}(Q_T))$.
Moreover, by the Young's inequality and the fact that $\gamma \in L^\infty(Q_T)$ we have

$$\begin{aligned} \int_{Q_\tau} \Phi(x,t,T_K(u_n))\nabla T_K(u_n)\,dx\,dt &\leq C_\gamma \int_{Q_\tau} P(x,|T_K(u_n)|)\,dx\,dt \\ &+ C_\gamma \int_{Q_\tau} P(x,\nabla T_K(u_n))\,dx \end{aligned} \tag{62}$$

where $C_\gamma = ||\gamma||_{L^\infty(Q_T)}$.
From the Remark 2.7 and (43)we therefore get

$$\begin{aligned} &\int_{Q_\tau} P(x,T_K(u_n))\,dx\,dt = \int_{\{(x,t)\in Q_\tau;|T_K(u_n)|\leq t_0\}} P(x,T_K(u_n))\,dx\,dt \\ &+ \int_{\{(x,t)\in Q_\tau;|T_K(u_n)|\geq t_0\}} P(x,T_K(u_n))\,dx\,dt \\ &\leq \int_{\{(x,t)\in Q_\tau;|T_K(u_n)|\leq t_0\}} ess\sup_{x\in\Omega} P(x,|t_0|)\,dx\,dt \\ &+ \int_{\{(x,t)\in Q_\tau;|T_K(u_n)|\geq t_0\}} M(x,\varepsilon|T_K(u_n)|)\,dx\,dt \\ &\leq R_3 + \int_{Q_\tau} M(x,\varepsilon|T_K(u_n)|)\,dx\,dt. \end{aligned} \tag{63}$$

Using the same technics as above, one has

$$\int_{Q_\tau} P(x,|\nabla T_K(u_n)|)\,dx\,dt \leq R_4 + \int_{Q_\tau} M(x,\varepsilon|\nabla T_K(u_n)|)\,dx\,dt. \tag{64}$$

Hence

$$\begin{aligned}&\int_{Q_\tau}\Phi(x,t,T_K(u_n))\nabla T_K(u_n)\,dx\,dt\\&\le C_\gamma(R_3+R_4)+C_\gamma\int_{Q_\tau}M(x,\varepsilon|T_K(u_n)|)\,dx\,dt\\&+C_\gamma\int_{Q_\tau}M(x,\varepsilon|\nabla T_K(u_n)|)\,dx\,dt,\end{aligned}\tag{65}$$

where R_3 and R_4 are constants not depending on K and n. By choosing $\varepsilon=\frac{\alpha}{2C_\gamma+\alpha}$ and convexity of the function M we get

$$\begin{aligned}&\int_{Q_\tau}\Phi(x,t,T_K(u_n))\nabla T_K(u_n)\,dx\,dt\\&\le C_\gamma(R_3+R_4)+\tfrac{\alpha}{2}\int_{Q_\tau}M(x,|T_K(u_n)|)\,dx\,dt\\&+\tfrac{\alpha}{2}\int_{Q_\tau}M(x,|\nabla T_K(u_n)|)\,dx\,dt.\end{aligned}\tag{66}$$

From (28), (61) and (66) we deduce that

$$\int_{Q_\tau}M(x,|\nabla T_K(u_n)|)\,dx\,dt\le CK\text{ for }K\ge 1.\tag{67}$$

Where C is a positive constant independent of K and n.
We prove (57). Indeed, it result from (28) and (67) that

$$\text{meas}\{(x,t)\in Q_T;|u_n|>K\}\le\frac{CK}{\inf\limits_{x\in\Omega}M(x,K)}.\tag{68}$$

Let tending K to infinity. We deduce:

$$\lim_{K\to\infty}\text{meas}\{(x,t)\in Q_T;|u_n|>K\}=0.\tag{69}$$

Then we conclude that there exists some $v_K\in W_0^{1,x}L_M(Q_T)$ such that

$$T_K(u_n)\rightharpoonup v_K\text{ weakly in }W_0^{1,x}L_M(Q_T)\text{ for }\sigma(\Pi L_M,\Pi E_{\overline{M}}).\tag{70}$$

Let $\varepsilon>0$, since (57), (70) and the fact $T_K(u_n)$ is a Cauchy sequence in measure, there exists some $K_\varepsilon>0$ such that meas$\{(x,t)\in Q_T;|u_n-u_m|>\lambda\}$ for all $n,m>N_0(K_\varepsilon,\lambda)$. This proves that $(u_n)_n$ is a Cauchy sequence in measure in Q_T thus converges almost everywhere to some measurable function u.
We conclude that there exists some $u\in W_0^{1,x}L_M(Q_T)$ such that

$$T_K(u_n)\rightharpoonup T_K(u)\text{ weakly in }W_0^{1,x}L_M(Q_T)\ ,\tag{71}$$

for $\sigma(\Pi L_M,\Pi E_{\overline{M}})$
Next, if we multiply the approximation equation (40) by $\theta_K'(t)$, where $\theta_K(.)$ is a $\mathcal{C}^2(\mathbb{R}$ nondecreasing function such that $\theta_K(t)=t$ for $|t|\le\frac{K}{2}$ and $\theta_K(t)=K$ for $|t|\ge K$, we obtain

$$\begin{aligned}&\tfrac{\partial\theta_k(u_n)}{\partial t}=\text{div}\Big(a(x,t,u_n,\nabla u_n)\theta_k'(u_n)\Big)\\&-a(x,t,u_n,\nabla u_n)\theta_k''(u_n)\nabla u_n\\&+\text{div}\Big(\theta_k'(u_n)\Phi(x,t,u_n)\Big)\\&-\Phi(x,t,u_n)\theta_k''(u_n)\nabla u_n+f_n\theta_k'(u_n),\end{aligned}\tag{72}$$

in the sense of distributions.
Due to(26) and the fact that $T_K(u_n)$ is bounded in $W_0^{1,x}L_M(Q_T)$, the term
-div$\Big(a(x,tu_n,\nabla u_n)\theta_K'(u_n)\Big)+a(x,t,u_n,\nabla u_n)\theta_K''(u_n)+f_n\theta_K'(u_n)$ is bounded in $W^{-1}L_{\overline{M}}(Q_T)$. Furthermore, we have supp(θ_K') and supp(θ_K'') are both in $[-K,K]$, which gives

$$\begin{aligned}&|\int_{Q_T}\theta_K''(u_n)\Phi(x,t,u_n)\nabla u_n\,dx\,dt|\\&\le\|\theta_K''\|_{L^\infty}\int_{Q_T}|\Phi(x,t,T_K(u_n)||T_K(u_n)|\,dx\,dt,\end{aligned}$$

by (29), $\gamma\in L^\infty(Q_T)$ and the Young's inequality it follows that

$$\begin{aligned}&|\int_{Q_T}\theta_K''(u_n)\Phi(x,t,u_n)\nabla u_n\,dx\,dt|\le\|\theta_K''\|_{L^\infty}\|\gamma\|_{L^\infty(Q_T)}\\&\times\Big[\int_{Q_T}P(x,|\nabla T_K(u_n)|)\,dx\,dt+\int_{Q_T}P(x,|\nabla T_K(u_n)|)\,dx\,dt\Big].\end{aligned}\tag{73}$$

By applying the same Technics as in the proof of Lemma 5.4, we prove that $\theta_K''(u_n)\Phi(x,t,u_n)\nabla u_n$ is bounded in $L^1(Q_T)$. In the same way, we show that div$\Big(\theta_k'(u_n)\Phi(x,t,u_n)\Big)$ is bounded in $W^{-1,x}L_M(Q_T)$.
Hence all bove implies that

$$\frac{\partial\theta_k(u_n)}{\partial t}\text{ is bounded in }W^{-1,x}L_{\overline{M}}(Q_T)+L^1(Q_T.\tag{74}$$

Proceeding as in [35] and using Corollary 3.10, we easily show that there exists a mesurable function $u\in L^\infty(0,T;L^1(\Omega))$ such that for every $K>0$

$$T_K(u_n)\rightharpoonup T_k(u)\text{ weakly in }W^{1,x}L_M(Q_T)\text{ for }\sigma(\mathit{\Pi} L_M,\mathit{\Pi} E_{\overline{M}})\tag{75}$$

and

$$T_K(u_n)\to T_k(u)\ \text{ strongly in }\ L^1(Q_T)\ \text{ and a.e in }Q_T\tag{76}$$

Now, we prove the following lemma
Lemma 5.6: Let u_n be a solution of the approximate problem (40), then for all $K\ge 0$,

$$\Big(a(x,t,T_K(u_n),\nabla T_K(u_n))\Big)_n\text{ is bounded in }(L_{\overline{M}}(Q_T))^N.\tag{77}$$

Proof 5.7: Let $\varphi\in(E_M(Q_T))^N$ be arbitrary. In view of the monotonicity of a, one easily has

$$\Big(a\Big(x,t,u_n,\nabla u_n\Big)-a\Big(x,t,u_n,\varphi\Big)\Big)\Big(\nabla u_n-\varphi\Big)\ge 0.\tag{78}$$

Hence

$$\begin{aligned}&\int_{\{|u_n|\le K\}}a\Big(x,t,u_n,\nabla u_n\Big)\varphi\,dx\,dt\\&\le\int_{\{|u_n|\le K\}}a\Big(x,t,u_n,\nabla u_n\Big)\nabla u_n\,dx\,dt\\&+\int_{\{|u_n|\le K\}}a\Big(x,t,u_n,\varphi\Big)\Big(\varphi-\nabla u_n\Big)\,dx\,dt.\end{aligned}\tag{79}$$

Using (26) and since $T_K(u_n)$ is bounded in $W_0^{1,x}L_M(Q_T)$, one easily deduces that

$$\int_{Q_T} a\Big(x,t,T_K(u_n),\nabla T_K(u_n)\Big)\nabla T_K(u_n)\,dx\,dt \le CK_1. \tag{80}$$

Combining the fact that $T_K(u_n)$ is bounded in $W_0^{1,x}L_M(Q_T)$, (79) and (80), we get

$$\int_{Q_T} a\Big(x,t,T_K(u_n),\nabla T_K(u_n)\Big)\varphi\,dx\,dt \le CK_2. \tag{81}$$

Hence, thanks the Banach-Steinhaus Theorem, the sequence $\Big(a(x,t,T_K(u_n),\nabla T_K(u_n))\Big)_n$ is a bounded in $(L_{\overline{M}}(Q_T))^N$, thus up to a sub-sequence

$$a\Big(x,t,T_K(u_n),\nabla T_K(u_n)\Big) \rightharpoonup l_K \quad \text{in} \quad (L_{\overline{M}}(Q_T))^N \tag{82}$$

for $\sigma(\Pi L_{\overline{M}},\Pi E_M)$, for some $l_K \in (L_{\overline{M}}(Q_T))^N$.

Step 3 : Mudular convergence of the gradient.

This step is devoted to introduce for $K \ge 0$ fixed, a time regularization $w^i_{\mu,j}$ of the function $T_K(u)$.

We first introduce two smooth sequences, namely, $(v_j) \subset \mathcal{D}(Q_T)$ such that $v_j \to u$ in $W_0^{1,x}L_M(Q_T)$ for the modular convergence and almost everywhere in Q_T, and $(\psi_i) \subset \mathcal{D}(\Omega)$ which converges strongly to u_0 in $L^2(\Omega)$ and such that $\|\psi_i\|_{L^2(\Omega)} \le 2\|u_0\|_{L^2(\Omega)}$, for all $i \ge 1$. For a fixed positive real number K, we consider the truncation function at height K, T_K. Then, for every $K,\mu > 0$ and $i,j \in \mathbb{N}$, we introduce the function $w^i_{\mu,j} \in W_0^{1,x}L_M(Q_T)$ (to simplify the notation, we drop out the index K) defined as $w^i_{\mu,j} = T_K(v_j)_\mu + e^{-\mu t}T_K(\psi_i)$, where $T_K(v_j)_\mu$ is the mollification with respect to time of $T_K(v_j)$ given in (17). From Lemma (3.1), we know that

$$\frac{\partial w^i_{\mu,j}}{\partial t} = \mu(T_K(v_j)-w^i_{\mu,j}),\ w^i_{\mu,j}(\cdot,0) = T_K(\psi_i),\ |w^i_{\mu,j}| \le K \tag{83}$$

a.e in Q_T,

$$w^i_{\mu,j} \to w^i_\mu \overset{\text{def}}{=} T_K(u)_\mu + e^{-\mu t}T_K(\psi_i) \text{ in } W_0^{1,x}L_M(Q_T), \tag{84}$$

for the modular convergence as $j \to \infty$.

$$T_K(u)_\mu + e^{-\mu t}T_K(\psi_i) \to T_K(u) \text{ in } W_0^{1,x}L_M(Q_T), \tag{85}$$

for the modular convergence as $\mu \to \infty$.

We will establish the following proposition.

Proposition 5.8: Let u_n be a solution of the approximate problem (25)-(29). Then, for any $K \ge 0$:

$$\nabla u_n \to \nabla u \quad \text{a.e. in} \quad Q_T, \tag{86}$$

$$a\Big(x,t,T_K(u_n),\nabla T_K(u_n)\Big) \rightharpoonup a\Big(x,t,T_K(u),\nabla T_K(u)\Big) \tag{87}$$

weakly in $(L_{\overline{M}}(Q_T))^N$,

$$M(|\nabla T_K(u_n)|) \to M(|\nabla T_K(u)|) \quad \text{strongly in} \quad L^1(Q_T), \tag{88}$$

as n tends to $+\infty$.

Let us consider the function h_m defined on $\mathbb{R}$ by:

$$h_m(s)\begin{cases} 1 & \text{if } |s| \le m \\ -|s|+m+1 & \text{if } m \le |s| \le m+1 \\ 0 & \text{if } |s| \ge m+1, \end{cases}$$

for any $m \ge K$.

Using the admissible test function $\varphi^{\mu,i}_{n,j,m} = (T_K(u_n) - w^\mu_{i,j})h_m(u_n)$ as test function in (40) leads to

$$\langle \frac{\partial u_n}{\partial t}, \varphi^{\mu,i}_{n,j,m}\rangle + \int_{Q_T} a(x,t,u_n,\nabla u_n) \tag{89}$$

$$\times(\nabla T_K(u_n) - \nabla w^\mu_{i,j})h_m(u_n)\,dx\,dt.$$

$$+\int_{Q_T} a(x,t,u_n,\nabla u_n)(T_K(u_n) - w^\mu_{i,j})\nabla u_n h'_m(u_n)\,dx\,dt$$

$$+\int_{\{m\le|u_n|\le m+1\}} \Phi(x,t,u_n)\nabla u_n h'_m(u_n)(T_K(u_n)-w^\mu_{i,j})\,dx\,dt$$

$$+\int_{Q_T} \Phi(x,t,u_n)h_m(u_n)(\nabla T_K(u_n) - \nabla w^\mu_{i,j})\,dx\,dt$$

$$= \int_{Q_T} f_n\varphi^{\mu,i}_{n,j,m}\,dx\,dt.$$

Denoting by $\epsilon(n,j,\mu,i)$ any quantity such that,

$$\lim_{i\to\infty}\lim_{\mu\to\infty}\lim_{j\to\infty}\lim_{n\to\infty}\epsilon(n,j,\mu,i) = 0.$$

By the definition of the sequence $w^\mu_{i,j}$, we can establish the following lemma.

Lemma 5.9: Let $\varphi^{\mu,i}_{n,j,m} = (T_K(u_n)-w^\mu_{i,j})h_m(u_n)$, we have for any $K \ge 0$:

$$\langle \frac{\partial u_n}{\partial t}, \varphi^{\mu,i}_{n,j,m}\rangle \ge \epsilon(n,j,\mu,i), \tag{90}$$

where $\langle,\rangle$ denotes the duality pairing between $L^1(Q_T) + W^{-1,x}L_{\overline{M}}(Q_T)$ and $L^\infty(Q_T) \cap W_0^{1,x}L_M(Q_T)$.

Proof 5.10: Using the same techniques as in Orlicz space (see [6]), we can easily get the result.

Now, we turn to complete the proof of Proposition 5.8., we prove below the following results for any fixed $K \ge 0$.

$$\int_{Q_T} f_n\varphi^{\mu,i}_{n,j,m}\,dx\,dt = \epsilon(n,j,\mu). \tag{91}$$

$$\int_{Q_T} \Phi(x,t,u_n)h_m(u_n)(\nabla T_K(u_n)-\nabla w^\mu_{i,j})\,dx\,dt = \epsilon(n,j,\mu), \tag{92}$$

$$\int_{\{m\le|u_n|\le m+1\}} \Phi(x,t,u_n)\nabla u_n h'_m(u_n)(T_K(u_n)-w^\mu_{i,j})\,dx\,dt = \epsilon(n,j,\mu \tag{93}$$

$$\int_{Q_T} a(x,t,u_n,\nabla u_n)\nabla u_n h'_m(u_n)(T_K(u_n)-w^\mu_{i,j})\,dx\,dt \le \epsilon(n,j,\mu,m). \tag{94}$$

$$\int_{Q_T}\Big[a\Big(x,t,T_K(u_n),\nabla T_K(u_n)\Big)-a\Big(x,t,T_K(u_n),\nabla T_K(u)\chi_s\Big)\Big] \times\Big[\nabla T_K(u_n)-\nabla T_K(u)\chi_s\Big]\,dx\,dt\le\epsilon(n,j,\mu,m,s). \tag{95}$$

Proof of (91) : By the almost everywhere convergence of u_n, we have $(T_K(u_n)-w^{\mu}_{i,j})h_m(u_n)$ converges to $(T_K(u)-w^{\mu}_{i,j})h_m(u)$ in $L^\infty(Q_T)$ weak-* and then,

$$\int_{Q_T} f_n(T_K(u_n)-w^{\mu}_{i,j})h_m(u_n)\,dx\,dt \to \int_{Q_T} f(T_K(u)-w^{\mu}_{i,j})h_m(u)\,dx\,dt.$$

So that,

$$(T_K(u)-w^{\mu}_{i,j})h_m(u)\to(T_K(u)-T_K(u)_\mu-e^{-\mu t}T_K(\psi_i))$$

in $L^\infty(Q_T)$ weak-* as $j\to\infty$, and also

$$(T_K(u)-T_K(u)_\mu-e^{-\mu t}T_K(\psi_i))\to 0$$

in $L^\infty(Q_T)$ weak-* as $\mu\to+\infty$. Then, we deduce that,

$$\int_{Q_T} f_n(T_K(u_n)-w^{\mu}_{i,j})h_m(u_n)\,dx\,dt=\epsilon(n,j,\mu). \tag{96}$$

Proof of (92) and (93): For n large enough, we have

$$\Phi(x,t,u_n)h_m(u_n)=\Phi(x,t,T_{m+1}(u_n))h_m(T_{m+1}(u_n)) \tag{97}$$

a.e in Q_T.

In order to prove (92) and (93), we will apply Lemma 2.9, Let remark that $P\ll M\Leftrightarrow \overline{M}\ll\overline{P}$ (see [25]). Thus we need only to show that $\Phi(x,t,T_{m+1}(u_n))$ converge to $\Phi(x,t,T_{m+1}(u))$ with respect to the modular convergence in $(L_{\overline{P}}(Q_T))^N$ to get the desired result.

Indeed, we put $M_n=\overline{P}\Big(x,\frac{\Phi(x,t,T_{m+1}(u_n))-\Phi(x,t,T_{m+1}(u))}{\mu}\Big)$. we have that Φ is a Carathéodory function and using the pointwise convergence of u_n we get that $\Phi(x,t,T_{m+1}(u_n))\to\Phi(x,t,T_{m+1}(u))$ a.e in Q_T as $n\to\infty$, then since $\overline{P}(0)=0$, one has

$$M_n=\overline{P}\Big(x,\frac{\Phi(x,t,T_{m+1}(u_n))-\Phi(x,t,T_{m+1}(u))}{\mu}\Big)\to 0, \tag{98}$$

a.e in Q_T as $n\to\infty$.

By the convexity of $\overline{P}$, for μ and n large enough and by (29), we obtain

$$\begin{aligned} M_n&=\overline{P}\Big(x,\tfrac{\Phi(x,t,T_{m+1}(u_n))-\Phi(x,t,T_{m+1}(u))}{\mu}\Big)\\ &\le\tfrac{C_\gamma}{\mu}\overline{P}(x,\overline{P}_x^{-1}P(x,|T_{m+1}(u_n)|)|)\\ &+\tfrac{C_\gamma}{\mu}\overline{P}(x,\overline{P}_x^{-1}P(x,|T_{m+1}(u)|)|)\\ &\le\tfrac{2C_\gamma}{\mu}ess\sup_{x\in\Omega}P(x,m+1)=C_m\quad\text{a.e. in } Q_T. \end{aligned} \tag{99}$$

By Remark 2.7 we have $C_m\in L^1(Q_T)$. Then, using (98), (99) and by Lebesgue's dominated convergence theorem, we obtain

$$\int_{Q_T} M_n\,dx\to 0\qquad\text{as } n \text{ goes to infinity.} \tag{100}$$

Hence

$$\Phi(x,t,T_{m+1}(u_n))\to\Phi(x,t,T_{m+1}(u)) \tag{101}$$

with repect to the modular convergence in $L_{\overline{P}}(Q_T)$ as $n\to+\infty$. By appling Lemma 2.9. we obtain $\Phi(x,t,T_{m+1}(u_n))\to\Phi(x,t,T_{m+1}(u))$ in $(E_{\overline{M}}(Q_T))^N$.

Then by virtue of, $\nabla T_K(u_n)\rightharpoonup\nabla T_K(u)$ weakly in $(L_M(Q_T))^N$, then

$$\int_{Q_T}\Phi(x,t,u_n)h_m(u_n)(\nabla T_K(u_n)-\nabla w^{\mu}_{i,j})\,dx\,dt \to\int_{Q_T}\Phi(x,t,u)h_m(u)(\nabla T_K(u)-\nabla w^{\mu}_{i,j})\,dx\,dt \tag{102}$$

as $n\to+\infty$.

In the other hand, by using the modular convergence of $w^{\mu}_{i,j}$ as $j\to+\infty$ and letting μ tends to infinity, we get (92).

Now we turn to prove (93).

First, remark for $n\ge m+1$ we have that

$$\nabla u_n h_m'(u_n)=\nabla T_{m+1}(u_n)\quad\text{a.e in}\quad Q_T. \tag{103}$$

By the almost everywhere convergence of u_n, we have $(T_K(u_n)-w^{\mu}_{i,j})$ converges to $(T_K(u)-w^{\mu}_{i,j})$ in $L^\infty(Q_T)$ weak-* and since the sequence $(\Phi(x,t,T_{m+1}(u_n)))_n$ converges strongly in $E_{\overline{M}}(Q_T)$ then,

$$\Phi(x,t,T_{m+1}(u_n))(T_K(u_n)-w^{\mu}_{i,j})\to\Phi(x,t,T_{m+1}(u))(T_K(u)-w^{\mu}_{i,j})$$

converges strongly in $E_{\overline{M}}(Q_T)$ as n goes to $+\infty$.

Using again the fact that, $\nabla T_{m+1}(u_n)\rightharpoonup\nabla T_{m+1}(u)$ weakly in $(L_M(Q_T))^N$ as n tends to $+\infty$ we obtain

$$\int_{\{m\le|u_n|\le m+1\}}\Phi(x,t,u_n)\nabla u_n h_m'(u_n)(T_K(u_n)-w^{\mu}_{i,j})\,dx\,dt \to\int_{\{m\le|u|\le m+1\}}\Phi(x,t,u)\nabla u(T_K(u)-w^{\mu}_{i,j})\,dx\,dt, \tag{104}$$

as n tends to $+\infty$.

By using the modular convergence of $w^{\mu}_{i,j}$ as $j\to+\infty$ and letting μ tends to infinity, we get (93).

Proof of (94): Concerning the third term of the right hand side of (89) we obtain that

$$\int_{\{m\le|u_n|\le m+1\}}a(x,t,u_n,\nabla u_n)\nabla u_n h_m'(u_n)(T_K(u_n)-w^{\mu}_{i,j})\,dx\,dt \le 2K\int_{\{m\le|u_n|\le m+1\}}a(x,t,u_n,\nabla u_n)\nabla u_n\,dx\,dt. \tag{105}$$

Then by (77). we deduce that,

$$\int_{\{m\le|u_n|\le m+1\}}a(x,t,u_n,\nabla u_n)\nabla u_n h_m'(u_n)(T_K(u_n)-w^{\mu}_{i,j})\,dx\,dt \tag{106}$$

$\le\epsilon(n,\mu,m)$. which is the desired results.

Proof of (95): By means of (89)-(94), we obtain

$$\int_{Q_T}a(x,t,u_n,\nabla u_n)(\nabla T_K(u_n)-\nabla w^{\mu}_{i,j})h_m(u_n)\,dx\,dt\le\epsilon(n,\mu,m). \tag{107}$$

Using the same techniques as [24], we obtain

$$\lim_{s\to\infty}\lim_{n\to\infty}\int_{Q_T}\Big[a\Big(x,t,T_K(u_n),\nabla T_K(u_n)\Big) -a\Big(x,t,T_K(u_n),\nabla T_K(u)\chi_s\Big)\Big] \times\Big[\nabla T_K(u_n)-\nabla T_K(u)\chi_s\Big]\,dx\,dt=0. \tag{108}$$

This implies by the Lemma 4.1., the desired statement and hence the proof of Proposition 5.8. is achieved.

Step 4 : Passing to the limit

Let $v\in W^{1,x}L_M(Q_T)\cap L^\infty$ such that $\frac{\partial v}{\partial t}$ belongs to $W^{-1,x}L_{\overline{M}}(Q_T)+L^1(Q_T)$, there exists a prolongation $\overline{v}=v$ on Q_T, $\overline{v}\in W^{1,x}L_M(\Omega\times\mathbb{R})\cap L^1(\Omega\times\mathbb{R})\cap L^\infty(\Omega\times\mathbb{R})$, and

$$\frac{\partial v}{\partial t}\in W^{-1,x}L_{\overline{M}}(\Omega\times\mathbb{R})+L^1(\Omega\times\mathbb{R}).$$

There exists also a sequence $(\omega_j)\subset\mathcal{D}(\Omega\times\mathbb{R})$ such that

$$\begin{array}{ll}\omega_j\to\overline{v}\ \text{ in } & W_0^{1,x}L_M(\Omega\times\mathbb{R}),\ \text{ and}\\ \frac{\partial\omega_j}{\partial t}\to\frac{\partial\overline{v}}{\partial t}\ \text{ in } & W^{-1,x}L_{\overline{M}}(\Omega\times\mathbb{R})+L^1(\Omega\times\mathbb{R}).\end{array} \tag{109}$$

for the modular convergence and $||\omega_j||_{\infty,Q_T}\le(N+2)|v||_{\infty,Q_T}$ (see [2]).

Now, let us take $T_K(u_n-\omega_j)_{\chi_{(0,\tau)}}$ as a test function in (40), thus for every $\tau\in[0,T]$, we get

$$\begin{aligned}&\langle\frac{\partial u_n}{\partial t},T_K(u_n-\omega_j)\rangle_{Q_\tau}\\&+\int_{Q_\tau}a(x,t,T_{\widehat{K}}(u_n),\nabla T_{\widehat{K}}(u_n))\nabla T_K(u_n-\omega_j)\,dx\,dt\\&+\int_{Q_\tau}\Phi(x,t,T_{\widehat{K}}(u_n))\nabla T_K(u_n-\omega_j)\,dx\,dt\\&=\int_{Q_\tau}f_nT_K(u_n-\omega_j)\,dx\,dt,\end{aligned} \tag{110}$$

where $\widehat{K}=K+C||v||_{\infty,Q_T}$, which implies

$$\begin{aligned}&\langle\frac{\partial u_n}{\partial t},T_K(u_n-\omega_j)\rangle_{Q_\tau}\\&+\int_{Q_\tau\cap\{|u_n-\omega_j|\le K\}}a(x,t,T_{\widehat{K}}(u_n),\nabla T_{\widehat{K}}(u_n))\nabla u_n\,dx\\&-\int_{Q_\tau\cap\{|u_n-\omega_j|\le K\}}a(x,t,T_{\widehat{K}}(u_n),\nabla T_{\widehat{K}}(u_n))\nabla\omega_j\,dx\\&+\int_{Q_\tau}\Phi(x,t,T_{\widehat{K}}(u_n))\nabla T_K(u_n-\omega_j)\,dx\,dt\\&=\int_{Q_\tau}f_nT_K(u_n-\omega_j)\,dx\,dt.\end{aligned} \tag{111}$$

By Fatou's lemma and the fact that

$$a(x,t,T_{\widehat{K}}(u_n),\nabla T_{\widehat{K}}(u_n))\rightharpoonup a(x,t,T_{\widehat{K}}(u),\nabla T_{\widehat{K}}(u))$$

weakly in $(L_{\overline{M}}(Q_T))^N$ for $\sigma(\Pi L_{\overline{M}},\Pi E_M)$, one easily sees that

$$\begin{aligned}&\int_{Q_\tau\cap\{|u_n-\omega_j|\le K\}}a(x,t,T_{\widehat{K}}(u_n),\nabla T_{\widehat{K}}(u_n))\nabla u_n\,dx\\&-\int_{Q_\tau\cap\{|u_n-\omega_j|\le K\}}a(x,t,T_{\widehat{K}}(u_n),\nabla T_{\widehat{K}}(u_n))\nabla\omega_j\,dx\\&\ge\int_{Q_\tau\cap\{|u-\omega_j|\le K\}}a(x,t,T_{\widehat{K}}(u),\nabla T_{\widehat{K}}(u))\nabla u\,dx\\&-\int_{Q_\tau\cap\{|u-\omega_j|\le K\}}a(x,t,T_{\widehat{K}}(u),\nabla T_{\widehat{K}}(u))\nabla\omega_j\,dx.\end{aligned} \tag{112}$$

As in (98), we obtain $\Phi(x,t,T_{\widehat{K}}(u_n))\to\Phi(x,t,T_{\widehat{K}}(u))$ in $E_{\overline{M}}(Q_T)$ as $n\to+\infty$ and using the fact that $\nabla T_K(u_n-\omega_j)\rightharpoonup\nabla T_K(u-\omega_j)$ in $L_M(Q_T)$, as $n\to+\infty$, we can easy see that

$$\int_{Q_\tau}\Phi(x,t,T_{\widehat{K}}(u_n))\nabla T_K(u_n-\omega_j)\,dx\,dt \to\int_{Q_\tau}\Phi(x,t,T_{\widehat{K}}(u))\nabla T_K(u-\omega_j)\,dx\,dt. \tag{113}$$

Since $T_K(u_n-\omega_j)\to T_K(u-\omega_j)$ weakly* in L^∞ as $n\to+\infty$, we have

$$\int_{Q_\tau}f_nT_K(u_n-\omega_j)\,dx\,dt\to\int_{Q_\tau}fT_K(u-\omega_j)\,dx\,dt.$$

Turn now to see the first term of (110),

$$\begin{aligned}\langle\frac{\partial u_n}{\partial t},T_K(u_n-\omega_j)\rangle_{Q_\tau}=&\int_\Omega\Theta_K(u_n-\omega_j)dx\\&+\langle\frac{\partial\omega_j}{\partial t},T_K(u_n-\omega_j)\rangle_{Q_\tau}\\&-\int_\Omega\Theta_K(u_{n0}-\omega_j(0))dx.\end{aligned} \tag{114}$$

First, let see that $u_n\to u$ in $C([0,T];L^1(\Omega))$ (see [19]). Moreover, since $\Theta_K(u_n-\omega_j)(\tau)\le K|u_n(\tau)|+K|\omega_j(\tau)|$, we have by Lebesgue Theorem

$$\int_\Omega\Theta_K(u_n-\omega_j)(\tau)dx\to\int_\Omega\Theta_K(u-\omega_j)(\tau)dx,$$

as $n\to+\infty$. Then, we can pass to the limit in (114) as $n\to+\infty$ we obtain

$$\begin{aligned}\lim_{n\to+\infty}\langle\frac{\partial u_n}{\partial t},T_K(u_n-\omega_j)\rangle_{Q_\tau}=&\int_\Omega\Theta_K(u-\omega_j)dx\\&+\langle\frac{\partial\omega_j}{\partial t},T_K(u-\omega_j)\rangle_{Q_\tau}\\&-\int_\Omega\Theta_K(u_0-\omega_j(0))dx.\end{aligned} \tag{115}$$

Now, let n goes to infinity in (110), we get

$$\begin{aligned}
&\int_\Omega \Theta_K(u-\omega_j)dx + \langle \frac{\partial \omega_j}{\partial t}, T_K(u-\omega_j)\rangle_{Q_\tau} \\
&+ \int_{Q_\tau} a(x,t,u,\nabla u)\nabla T_K(u-\omega_j)\ dx\ dt \\
&+ \int_{Q_\tau} \Phi(x,t,u)\nabla T_K(u-\omega_j)\ dx\ dt \\
&\le \int_{Q_\tau} fT_K(u-\omega_j)\ dx\ dt \\
&+ \int_\Omega \Theta_K(u_0-\omega_j(0))dx.
\end{aligned} \tag{116}$$

By (109), as j tends to $+\infty$ we have

$$\langle \frac{\partial \omega_j}{\partial t}, T_K(u-\omega_j)\rangle_{Q_\tau} \to \langle \frac{\partial v}{\partial t}, T_K(u-v)\rangle_{Q_\tau}.$$

Moreover, for every $\tau \in [0,T]$, we have $||\omega_j - v(\tau)||_{L^1(\Omega)} \to 0$ as $j \to +\infty$. Therefore, we pass now to the limit as $j \to +\infty$ in (116), we get

$$\begin{aligned}
&\int_\Omega \Theta_K(u-v)dx + \langle \frac{\partial v}{\partial t}, T_K(u-v)\rangle_{Q_\tau} \\
&+ \int_{Q_\tau} a(x,t,u,\nabla u)\nabla T_K(u-v)\ dx\ dt \\
&+ \int_{Q_\tau} \Phi(x,t,u)\nabla T_K(u-v)\ dx\ dt \\
&\le \int_{Q_\tau} fT_K(u-v)\ dx\ dt + \int_\Omega \Theta_K(u_0-v(0))dx.
\end{aligned} \tag{117}$$

The proof of Theorem 5.2 is complete.

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