Solution of fuzzy differential equation by approximation of fuzzy number

Idris Bakhadach
Laboratory of Applied Mathematics
& Scientific Calculus,
Sultan Moulay Slimane University
idris.bakhadach@gmail.com

Hamid Sadiki
Laboratory of Applied Mathematics
& Scientific Calculus,
Sultan Moulay Slimane University
h.sadiki@usms.ma

Said Melliani
Laboratory of Applied Mathematics
& Scientific Calculus,
Sultan Moulay Slimane University
said.melliani@gmail.com

Lalla Saadia Chadli
Laboratory of Applied Mathematics
& Scientific Calculus,
Sultan Moulay Slimane University
sa.chadli@yahoo.fr

Abstract—In the present paper a definition of fuzzy algebra is presented, the condition of approximation of fuzzy number is proven. Finally the application to solve a fuzzy differential equation is given.

Index Terms—Fuzzy metric space, generalized fuzzy derivative, fuzzy algebra.

I. Introduction

Many scientific papers and many applications have proved that fuzzy set theory let us effectively model and transform imprecise information. It is not surprising that fuzzy numbers play an important role among all fuzzy sets since the predominant carrier of information are numbers. However, the crucial point in fuzzy modeling is to assign membership functions corresponding to fuzzy numbers that represent vague concepts and imprecise terms expressed often in a natural language. The representation does not only depend on the concept but also on the context in which it is used. But even for similar contexts, fuzzy numbers representing the same concept may vary considerably. Therefore, the problem of constructing meaningful membership functions is a difficult one and numerous methods for their construction have been described in the literature. All these methods may be classified into direct or indirect methods that involve one or multiple experts [13].

In practice, fuzzy intervals are often used to represent uncertain or incomplete information. An interesting problem is to approximate general fuzzy intervals by interval, triangular, and trapezoidal fuzzy numbers, so as to simplify calculations. Recently, many scholars investigated these approximations of fuzzy numbers. According to the different aspects of distance, these researches can be grouped into two classes: the Euclidean distance class [12] and the non-Euclidean distance class [8]. The autor in [10] give a necessary and sufficient conditions of linear operators which are preserved by interval, triangular, symmetric triangular, trapezoidal, or symmetric trapezoidal approximations of fuzzy numbers. In [11] presented a new nearest trapezoidal approximation operator preserving expected interval. But there is no work that has presented a stable part by multiplication, which is our goal in this paper with the approximation to one of whose element in order for example to give a sens of the solution of a differential equation whose contains the product of two specific fuzzy numbers.

This paper is organized as follows: After this introduction we present some concepts concerning the fuzzy metric space in section 2. The fuzzy algebra is defined in Section 3. A method of approximation is is discussed is Section 4, and we presented an application in the las section.

II. preliminaries

In this section, we present some definitions and introduce the necessary notation, which will be used throughout the paper.

We denote E^1 the class of function defined as follows:

$$E^1 = \Big\{u : \mathbb{R} \to [0,1], \quad u \quad \text{satisfies } (1-4) \text{ below}\Big\}$$

1) u is normal, i.e. there is a $x_0 \in \mathbb{R}$ such that $u(x_0) = 1$;
2) u is a fuzzy convex set;
3) u is upper semi-continuous;
4) u closure of $\{x \in \mathbb{R}^1, \quad u(x) > 0\}$ is compact

For all $\alpha \in (0,1]$ the α-cut of an element of E^1 is defined by

$$u^\alpha = \Big\{x \in \mathbb{R}, \ u(x) \geq \alpha\Big\}$$

By the previous properties we can write

$$u^\alpha = [\underline{u}(\alpha), \overline{u}(\alpha)]$$

The multiplication by a scalar is defined as follows

$$\lambda u(x) = \begin{cases} u(\frac{x}{\lambda}), & \lambda \neq 0 \\ \widetilde{0}, & \lambda = 0. \end{cases}$$

By the extension principal of Zadeh we have

$$\begin{aligned} (u+v)^\alpha &= u^\alpha + v^\alpha; \\ (\lambda u)^\alpha &= \lambda u^\alpha \end{aligned}$$

For all $u, v \in E^1$ and $\lambda \in \mathbb{R}$

The distance between two element of E^1 is given by (see [4])

$$d(u,v) = \sup_{\alpha \in (0,1]} \max\Big\{|\overline{u}(\alpha) - \overline{v}(\alpha)|, |\underline{u}(\alpha) - \underline{v}(\alpha)|\Big\}$$

The metric space (E^1, d) is complete, separable and locally compact and the following properties for metric d are valid:

1) $d(u+v, u+w) = d(u,v)$;
2) $d(\lambda u, \lambda v) = |\lambda| d(u,v)$;
3) $d(u+v, w+z) \leq d(u,w) + d(v,z)$;

Remark II.1. The space (E^1, d) is a linear normed space with $\|u\| = d(u,0)$.

Definition II.2. Let $u, v \in E^1$. We put $u^\alpha = \big[\underline{u}(\alpha), \overline{u}(\alpha)\big]$ and $v^\alpha = \big[\underline{v}(\alpha), \overline{v}(\alpha)\big]$. We define the product of u and v by

$$\begin{aligned}(u \odot v)^\alpha = \Big[&\min\big\{\underline{u}(\alpha)\underline{v}(\alpha), \underline{u}(\alpha)\overline{v}(\alpha), \overline{u}(\alpha)\underline{v}(\alpha), \overline{u}(\alpha)\overline{v}(\alpha)\big\}, \\ &\max\big\{\underline{u}(\alpha)\underline{v}(\alpha), \underline{u}(\alpha)\overline{v}(\alpha), \overline{u}(\alpha)\underline{v}(\alpha), \overline{u}(\alpha)\overline{v}(\alpha)\big\}\Big]\end{aligned}$$

Definition II.3. [6] The generalized Hukuhara difference of two fuzzy numbers $u, v \in E^1$ is defined as follows

$$u -_g v = w \Leftrightarrow \begin{cases} u = v + w \\ \text{or} \quad v = u + (-1)w \end{cases}$$

In terms of α-levels we have

$$\Big(u -_g v\Big)^\alpha = \Big[\min\{\underline{u}(\alpha) - \underline{v}(\alpha), \overline{u}(\alpha) - \overline{v}(\alpha)\}, \max\{\underline{u}(\alpha) - \underline{v}(\alpha), \overline{u}(\alpha) - \overline{v}(\alpha)\}\Big]$$

and the conditions for the existence of $w = u -_g v \in E^1$ are

$$\text{case (i)} \quad \begin{cases} \underline{w}(\alpha) = \underline{u}(\alpha) - \underline{v}(\alpha) \text{ and } \overline{w}(\alpha) = \overline{u}(\alpha) - \overline{v}(\alpha) \\ \text{with } \ \underline{w}(\alpha) \text{ increasing, } \ \overline{w}(\alpha) \text{ decreasing}, \underline{w}(\alpha) \leq \overline{w}(\alpha) \end{cases}$$

$$\text{case (ii)} \quad \begin{cases} \underline{w}(\alpha) = \overline{u}(\alpha) - \overline{v}(\alpha) \text{ and } \overline{w}(\alpha) = \underline{u}(\alpha) - \underline{v}(\alpha) \\ \text{with } \ \underline{w}(\alpha) \text{ increasing, } \ \overline{w}(\alpha) \text{ decreasing}, \underline{w}(\alpha) \leq \overline{w}(\alpha) \end{cases}$$

for all $\alpha \in [0,1]$.

Proposition II.4. [6]

$$\|u -_g v\| = d(u,v)$$

Since $\|.\|$ is a norm on E^1 and by the proposition (II.4) we have

Proposition II.5.

$$\|\lambda u -_g \mu u\| = |\lambda - \mu| \|u\|$$

Let $f : [a,b] \subset \mathbb{R} \to E^1$ a fuzzy-valued function. The α-level of f is given by

$$f(x,\alpha) = \Big[\underline{f}(x,\alpha), \overline{f}(x,\alpha)\Big], \ \forall x \in [a,b], \ \forall \alpha \in [0,1].$$

Definition II.6. [6] Let $x_0 \in (a,b)$ and h be such that $x_0 + h \in (a,b)$, then the generalized Hukuhara derivative of a fuzzy value function $f : (a,b) \to E^1$ at x_0 is defined as

$$\lim_{h \to 0} \Big\| \frac{f(x_0+h) -_g f(x_0)}{h} -_g f'_{gH}(x_0) \Big\| = 0 \qquad \text{(II.3)}$$

If $f_{gH}(x_0) \in E^1$ satisfying (II.3) exists, we say that f is generalized Hukuhara differentiable (gH-differentiable for short) at x_0.

Definition II.7. [6] Let $f : [a,b] \to E^1$ and $x_0 \in (a,b)$, with $\underline{f}(x,\alpha)$ and $\overline{f}(x,\alpha)$ both differentiable at x_0.
We say that

1) f is $[(i) - gH]$-differentiable at x_0 if

$$f'_{i,gH}(x_0) = \Big[\underline{f}'(x,\alpha), \overline{f}'(x,\alpha)\Big] \qquad \text{(II.4)}$$

2) f is $[(ii) - gH]$-differentiable at x_0 if

$$f'_{ii,gH}(x_0) = \Big[\overline{f}'(x,\alpha), \underline{f}'(x,\alpha)\Big] \qquad \text{(II.5)}$$

Theorem II.8. Let $f : J \subset \mathbb{R} \to E^1$ and $g : J \to \mathbb{R}$ and $x \in J$. Suppose that $g(x)$ is differentiable function at x and the fuzzy-valued function $f(x)$ is gH-differentiable at x. So

$$(fg)'_{gH} = (f'g)_{gH} + (fg')_{gH}$$

Proof. Using (II.5), for h enough small we get

$$\left\|\frac{f(x+h)g(x+h)-_g f(x)g(x)}{h}-_g((f'(x)g(x))_{gH}+(f(x)g'(x))_{gH})\right\|$$

$$=\left\|\frac{f(x+h)g(x+h)-_g f(x)g(x+h)+f(x)g(x+h)}{h}\right.$$
$$\left.\frac{-_g f(x)g(x)}{h}-_g((f'(x)g(x))_{gH}+(f(x)g'(x))_{gH})\right\|$$
$$=\left\|\frac{(f(x+h)-_g f(x))g(x+h)+f(x)(g(x+h)-_g g(x))}{h}\right.$$
$$\left.-_g((f'(x)g(x))_{gH}+(f(x)g'(x))_{gH})\right\|$$
$$\leq\left\|\frac{(f(x+h)-_g f(x))g(x+h)}{h}-_g((f'(x)g(x))_{gH})\right\|$$
$$+\left\|\frac{(f(x)(g(x+h)-_g g(x))}{h}-_g((f(x)g'(x))_{gH})\right\|$$
$$\leq\left\|\frac{(f(x+h)-_g f(x))}{h}g(x+h)-_g((f'(x)g(x))_{gH})\right\|$$
$$+\left\|f(x)\frac{((g(x+h)-_g g(x))}{h}-_g((f(x)g'(x))_{gH})\right\|$$

which complet the proof by passing to limit. □

Definition II.9. [9] Let $f:[a,b]\to E^1$. We say that $f(x)$ is fuzzy Riemann integrable to $I\in E^1$ if for any $\epsilon>0$, there exists $\delta>0$ such that for any division $P=\{[u,v];\xi\}$ with the norms $\Delta(P)<\delta$, we have

$$d\left(\sum_p^*(v-u)f(\xi),I\right)<\epsilon$$

where $\sum_p^*$ denote the fuzzy summation. We choose to write $I=\int_a^b f(x)dx$.

Theorem II.10. [6] If f is gH-differentiable with no switching point in the interval $[a,b]$ then we have

$$\int_a^b f(t)dt=f(b)-_g f(a)$$

III. Fuzzy algebra

In this section consider $\mathcal{P}$ the set of all $\mu\in E^1$ such that

$$\mu^\alpha=\big[-a^\alpha,a^\alpha\big],\quad\forall\alpha\in[0,1]$$

where $a\in[0,1]$

Lemma III.1. $(\mathcal{P},+)$ is a group.

Proof. First $\{0\}=[0,0]\in\mathcal{P}$.
Now let $\mu,\nu\in\mathcal{P}$ putting $\mu^\alpha=[-a^\alpha,a^\alpha]$ and $\nu^\alpha=[-b^\alpha,b^\alpha]$.
Using case (i) we get

$$\mu^\alpha-_g\nu^\alpha=\big[-a^\alpha-b^\alpha,a^\alpha+b^\alpha\big]$$
$$=\big[-c^\alpha,c^\alpha\big]$$

where $c=(a^\alpha+b^\alpha)^{\frac{1}{\alpha}}$.
So $\mu-_g\nu\in\mathcal{P}$, thus $(\mathcal{P},+)$ is a group. □

Lemma III.2. The product $\odot$ is stable on $\mathcal{P}$.

Proof. Let $\mu,\nu\in\mathcal{P}$ putting $\mu^\alpha=[-a^\alpha,a^\alpha]$ and $\nu^\alpha=[-b^\alpha,b^\alpha]$.
We have

$$(\mu\odot\nu)^\alpha=\big[-(ab)^\alpha,(ab)^\alpha\big]$$

so $\mu\odot\nu\in\mathcal{P}$. □

Theorem III.3. The quadruplet $(\mathcal{P},+,\odot,.)$ is an algebra.

Proof. First prove that the triplet $(\mathcal{P},+,.)$ is a vector space.
By lemmas III.1 and III.2 $(\mathcal{P},+)$ is a group ($-_g$ is the inverse of $+$) and stable by ..
Now let $\lambda\in\mathbb{R}^+$ and $\gamma\in\mathbb{R}^-$ and $\mu\in\mathcal{P}$, we have

$$(\lambda+\gamma).\mu=\lambda.\mu-_g(-\gamma).\mu$$

so $(\mathcal{P},.)$ is a vector space which implies the result. □

Definition III.4. The quadruplet $(\mathcal{P},+,\odot,.)$ is said a fuzzy algebra.

Definition III.5. Let $u,v\in E^1$. We put $u^\alpha=\big[\underline{u}(\alpha),\overline{u}(\alpha)\big]$ and $v^\alpha=\big[\underline{v}(\alpha),\overline{v}(\alpha)\big]$. we define

$$d_2(u,v)=\Big[\int_0^1(\overline{u}(\alpha)-\overline{v}(\alpha))^2d\alpha+\int_0^1(\underline{u}(\alpha)-\underline{v}(\alpha))^2d\alpha\Big]^{\frac{1}{2}}$$

By Brezis [7] $L^2(\mathbb{R})$ is a complet space it is easy to deduce the following proposition.

Proposition III.6. (E^1,d_2) is a complet metric space.

Now we define the following map

$$\langle.,.\rangle:\begin{cases}\mathcal{P}\times\mathcal{P}\to\mathbb{R}\\(\mu,\nu)\to\langle\mu,\nu\rangle=\int_0^1\underline{\mu}(\alpha)\underline{\nu}(\alpha)+\overline{\mu}(\alpha)\overline{\nu}(\alpha)d\alpha\end{cases}$$

Proposition III.7. The map $\langle.,.\rangle$ define an inner product on $\mathcal{P}$.

Proof. The linearity of integral show the bilinearity of the map, also the symmetry is clear, we have $\langle\mu,\mu\rangle\geq0$.
If $\langle\mu,\mu\rangle=0$ then $\underline{\mu}(\alpha)=\overline{\mu}(\alpha)=0$, a.e. on $[0,1]$ we get $\mu=0$. □

By III.6 we have

Proposition III.8. $(E^1,\langle.,.\rangle)$ is a Hilbert space.

Note that the norm associated to the inner product $\langle.,.\rangle$ is defined as follows

$$\|\mu\|_2=\sqrt{\int_0^1\underline{\mu}^2(\alpha)+\overline{\mu}^2(\alpha)d\alpha},\quad\forall\mu\in E^1.$$

Proposition III.9. $\mathcal{P}$ is a convex closed subspace of E^1, by respect the d_2 metric.

Proof. Let $\mu\in\overline{\mathcal{P}}$, then there exist $\mu_n\in\mathcal{P}$ such that $d_2(\mu_n,\mu)\to0$, By Brezis [7] theorem 4.9 page 94 there existe a subsequence μ_{n_k} of μ_n converge to μ, if we put $\mu_n^\alpha=\big[-a_n^\alpha,a_n^\alpha\big]$ for all $\alpha\in(0,1]$ then a_{n_k} converge to $a\in\mathbb{R}^+$, which implies that $\mu^\alpha=\big[-a^\alpha,a^\alpha\big]$, for all

$\alpha \in (0,1]$, by consequent $\mu \in \mathcal{P}$.
Now let $t \in [0,1]$ and $\mu \in \mathcal{P}$, we get

$$t\big[-a^{\alpha}, a^{\alpha}\big] + (1-t)\big[-b^{\alpha}, b^{\alpha}\big] = \big[-c^{\alpha}, c^{\alpha}\big]$$

where $c = \big(ta^{\alpha} + (1-t)b^{\alpha}\big)^{\frac{1}{\alpha}}$.
Hence $\mathcal{P}$ is convex. □

IV. Approximation

In this section we consider the set $E^1_{-}(a)$ defined as $E^1_{-} = \Big\{\mu \in E^1, \quad supp\mu = [-1,1], \quad \text{and } \mu^1 = [-a,a],$ for certain $a \in [0,1]\Big\}$.

Proposition IV.1. The set E^1_{-} is stable by the product $\odot$.

Proof. Let $\mu, \nu \in E^1_{-}(a)$. First note that $supp\mu \subset supp\mu \odot \nu$. In fact:
Let $x \in supp\mu$, then there is $(x_n)_n \subset \mathbb{R}$ such that $x_n \to x$ as $n \to \infty$, and $\mu(x_n) > 0$ which implies that $\mu(x) \geq 0$, we have two possibility if $\mu(x)\nu(x) > 0$ then $x \in supp(\mu)$ if not, $\mu(x)\nu(x) = 0$, since $supp(\nu) = supp(\mu) = [-1,1]$ then $x = -1$ or $x = 1$ but these two values are in $supp(\mu\nu)$.
Conversely: by the same it become that $supp(\mu \odot \nu) \subset supp(\mu)$.
Thus

$$supp(\mu \odot \nu) = \big[-1, 1\big]$$

By the definition of the product of two fuzzy number we have

$$(\mu \odot \nu)^1 = \big[-ab, ab\big]$$

where $\mu^{\alpha} = [-a^{\alpha}, a^{\alpha}]$ and $\nu^{\alpha} = [-b^{\alpha}, b^{\alpha}]$.
So

$$\mu \odot \nu \in \mathcal{P}.$$

□

Definition IV.2. Two fuzzy numbers μ and ν are approximately equal if and only if given a sufficiently small, we find that:

$$|\mu(x) - \nu(x)| \leq \epsilon, \quad \forall x \in \mathbb{R}.$$

we write $\mu \approx \nu$.

Proposition IV.3. Two fuzzy numbers μ and ν of $\mathcal{P}$ are approximately equal if and only if given a sufficiently small, we find that:

$$d_2\big(u, v\big) \leq \epsilon$$

Proof. We put $\mu^{\alpha} = \big[-a^{\alpha}, a^{\alpha}\big]$ and $\nu^{\alpha} = \big[-b^{\alpha}, b^{\alpha}\big]$, for all $\alpha \in [0,1]$. Where $a, b \in [0,1]$.
Since μ is increasing on $[-1,a]$ we get $\overline{\mu}(\alpha) = \mu^{-1}(x_{\alpha})$, also μ decreasing on $[a,1]$ then $\underline{\mu}(\alpha) = \mu^{-1}(y_{\alpha})$, this two functions are continuous at $\alpha \in [0,1]$.
If μ and ν are approximately equal, i.e.

$$\big|\mu(x) - \nu(x)\big| \leq \epsilon$$

we put $c = \min\{a, b\}$ thus

$$\big|\mu^{-1}(x) - \nu^{-1}(x)\big| \leq \epsilon$$

on $[-1,c]$, and

$$\big|\mu^{-1}(x) - \nu^{-1}(x)\big| \leq \epsilon$$

on $[c,1]$. Which implies that

$$\big|\underline{\mu}(\alpha) - \underline{\nu}(\alpha)\big| \leq \epsilon$$

and

$$\big|\overline{\mu}(\alpha) - \overline{\nu}(\alpha)\big| \leq \epsilon$$

so

$$d_2\big(\mu, \nu\big) \leq \epsilon.$$

Conversely:
Suppose thet $d_2\mu, \nu \leq \epsilon$, then $\overline{\mu}(\alpha) = \overline{\nu}(\alpha)$ and $\underline{\mu}(\alpha) - \underline{\nu}(\alpha)$ a.e.
By the continuity of $\alpha \to \overline{\mu}(\alpha) - \overline{\nu}(\alpha)$, we get $\overline{\mu}(\alpha) = \overline{\nu}(\alpha)$ and $\underline{\mu}(\alpha) - \underline{\nu}(\alpha)$, for all $\alpha \in [0,1]$. Using the idea of the previous part of this demonstration we have

$$\big|\mu(x) - \nu(x)\big| \leq \epsilon, \ \forall x \in \mathbb{R}.$$

Which complet the proof. □

Proposition IV.4. Any operation based on the extension principale of Zadeh preserve by the previous approximation.

Proof. Let μ, ν, μ' and ν' four fuzzy number such that

$$\mu \approx \mu' \quad \text{and} \quad \nu \approx \nu'.$$

We put $\xi = \mu * \nu$ and $\varsigma = \mu' * \nu'$.
Our goal is to prove that

$$\xi \approx \varsigma.$$

We can write

$$\xi(z) = \sup_{z=x*y} \min\{\mu(x), \nu(x)\}$$

and

$$\varsigma(z) = \sup_{z=x*y} \min\{\mu'(x), \nu'(x)\}$$

thus

$$\begin{aligned}\big|\xi(z) - \varsigma(z)\big| &= \Big|\sup_{z=x*y} \min\{\mu(x), \nu(x)\} - \sup_{z=x*y} \min\{\mu'(x), \nu'(x)\}\Big| \\ &\leq \Big|\sup_{z=x*y} \min\{\mu'(x)+\epsilon, \nu'(x)+\epsilon\} - \sup_{z=x*y} \min\{\mu'(x), \nu'(x)\}\Big| \\ &\leq \Big|\sup_{z=x*y} \min\{\mu'(x), \nu'(x)\} + \epsilon - \sup_{z=x*y} \min\{\mu'(x), \nu'(x)\}\Big| \\ &\leq \epsilon.\end{aligned}$$

which complet the proof. □

Theorem IV.5. Let for all $\mu \in E^1$, there exist $\nu \in \mathcal{P}$ such that

$$\mu \approx \nu.$$

Proof. Since $\big(E^1, \langle .,. \rangle\big)$ is a Hilbert space and $\mathcal{P}$ is a subset convex closed subset of E^1 then by theorem 5.2 in [7] there existe ν such that

$$d_2\big(\mu, \nu\big) \leq \epsilon$$

which implies $\mu \approx \nu$. □

Theorem IV.6. Let $\mu \in E^1_-$ and $\nu \in \mathcal{P}$ such that $\nu^\alpha = \big[-a^\alpha, a^\alpha\big]$.
A sufficient condition for $\mu \approx \nu$ is:

$$\max_{x\in J} \max\big\{\,|\mu(x-D)-\mu(x)|\,,|\mu(x+D)-\mu(x)|\,\big\} \leq \epsilon$$

where

$$\begin{cases} D = \max_{\alpha\in(0,1]} \big|a^\alpha - \overline{\mu}(\alpha)\big|, & \text{if} \quad J=[a,1] \\ D = \max_{\alpha\in(0,1]} \big|a^\alpha + \underline{\mu}(\alpha)\big|, & \text{if} \quad J=[-1,a] \end{cases}$$

Proof. For all $x\in[-1,a]$ it is clear that

$$x - D \leq \mu^{-1}(x) \leq x + D$$

since μ is an increasing function on $[-1,a]$ then

$$\mu(x-D) \leq x \leq \mu(x+D)$$

So that $\forall x \in [-1,a]$, if $\alpha = \mu(x)$ and $\alpha' = \nu(x)$, we get

$$\begin{aligned}\big|\alpha - \alpha'\big| &= \big|\mu(x) - \nu(x)\big| \\ &\leq \max\big\{\,|\mu(x-D)-\mu(x)|\,,|\mu(x+D)-\mu(x)|\,\big\}.\end{aligned}$$

in the same on $[a,1]$ we find the same result. □

Theorem IV.7. Let $u,v : I \to E^1$ tow derivative functions, where I is an intevalle of $\mathbb{R}$.
If $u(t)\approx v(t)$ for all $t\in I$, then $u'(t)\approx v'(t)$ for all $t \in I$.

Proof. If u and v are $[(i)-gH]$-differentiable or u and v are $[(ii)-gH]$-differentiable

$$u'_{i,gH}(x) = \Big[\underline{u}'(x,\alpha), \overline{u}'(x,\alpha)\Big]$$

and

$$v'_{i,gH}(x) = \Big[\underline{v}'(x,\alpha), \overline{v}'(x,\alpha)\Big]$$

or

$$u'_{i,gH}(x) = \Big[\overline{u}'(x,\alpha), \underline{u}'(x,\alpha)\Big]$$

and

$$v'_{i,gH}(x) = \Big[\overline{v}'(x,\alpha), \underline{v}'(x,\alpha)\Big]$$

By the proposition IV.3 we have $u(t)\approx v(t)$.
Now if u is $[(i)-gH]$-differentiable and v is $[(ii)-gH]$-differentiable then

$$u'_{i,gH}(x) = \Big[\underline{u}'(x,\alpha), \overline{u}'(x,\alpha)\Big]$$

and

$$v'_{i,gH}(x) = \Big[\overline{v}'(x,\alpha), \underline{v}'(x,\alpha)\Big]$$

this time also by the proposition IV.3 we have $u(t)\approx v(t)$. □

It is easy to note that

Corollary IV.8. If $u(t)\approx v(t)$ then $\int u(t) \approx \int v(t)$.

V. Application

In this section consider the following equation

$$y'(t) = \widetilde{a}y(t), \quad t \leq 0 \tag{V.1}$$

with $y\in E^1_-$ and $\widetilde{a}\in E^1_-$.
By approximation there exists $z, \widetilde{b} \in \mathcal{P}$ such that

$$z \approx y \quad \text{and} \quad \widetilde{b} \approx \widetilde{a}.$$

Now solving the following equation in $\mathcal{P}$

$$z' = \widetilde{b}z$$

by putting $b^\alpha = \big[-b^\alpha, b^\alpha\big]$ and $z(t) = \big[-(\eta(t))^\alpha, (\eta(t))^\alpha\big]$.
we get

$$(\eta'(t))^\alpha = b^\alpha(\eta(t))^\alpha, \quad \forall\alpha\in[0,1].$$

Which implies that

$$\eta(t) = ce^{bt}$$

where c is a constant.
Since $t \leq 0$, then $z\in E^1_-$. Using theorem IV.6 each fuzzy element μ verified the condition is a solution.

Remark V.1. By this method the solution of V.1 is not unique.

VI. Conclusions

This study makes it possible to give a meaning to the multiplication of two fuzzy numbers which makes us solve certain differential equations with uncertain initial values.

References

[1] R.P. Agarwal, S. Arshad, D. O'Regan, V. Lupulescu, Fuzzy fractional integral equations under compactness type condition, Fract. Calc. Appl. Anal. 15 (2012) 572–590.
[2] R.P. Agarwal, V. Lakshmikantham, J.J. Nieto, On the concept of solution for fractional differential equations with uncertainty, Nonlinear Anal. 72 (2010) 2859–2862.
[3] T. Allahviranloo, Z. Gouyandeh, A. Armand, A. Hasanoglu, On fuzzy solutions for heat equation based on generalized hukuhara differentiability, Fuzzy Sets Syst. 265 (2015) 1–23.
[4] G.A. Anastassiou, Fuzzy Mathematics: Approximation Theory. Studies in Fuzziness and Soft Computing, Springer, Berlin Heidelberg, 2010.
[5] A. Armand, Z. Gouyandeh, Fuzzy fractional integro-differential equations under generalized Caputo differentiability. Annals of Fuzzy Mathematics and Informatics Volume 10, No. 5, (November 2015), pp. 789798.
[6] B. Bede, L. Stefanini, Generalized differentiability of fuzzy-valued functions, Fuzzy Sets Syst. 230(0) (2013) 119–141.
[7] H. Brezis, Fonctional Analysis, Sobolev Spaces and Partial Differential Equations, Springer New York Dordrecht Heidelberg London, ISBN 978-0-387-70913-0.
[8] S. Chanas, On the interval approximation of a fuzzy number, Fuzzy Sets and Systems 122(2001) 353-356.
[9] M. Congxin, W. Ming, : Embedding problem of fuzzy number space: part III. Fuzzy Sets Syst. 46(2), 281–286 (1992)
[10] Chi-TsuenYeh, Existence of interval, triangular, and trapezoidal approximations of fuzzy numbers under a general condition, Fuzzy Sets and Systems, Article in press.
[11] P. Grzegorzewski, Edyta Mrowka, Trapezoidal approximations of fuzzy numbers, Fuzzy Sets and Systems 153 (2005) 115–135.
[12] P. Grzegorzewski, Nearest interval approximation of a fuzzy number, Fuzzy Sets and Systems 130(2002) 321-330.

[13] G.J. Klir, B.Yuan, Fuzzy sets and fuzzy logic, Theory and Applications, Prentice Hall, Englewood Cliffs, 1995.
[14] Zadeh L. A., Fuzzy sets, Information and Control, 8, (1965), 338–353.