Existence Result for a General Nonlinear Degenerate Elliptic Problems with Measure Datum in Weighted Sobolev Spaces

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***Abstract*—In this paper, we study the Dirichlet problem associated to the degenerate nonlinear elliptic equations**

$$\begin{cases} Lu(x)=\mu & \textbf{in } \Omega,\\ u(x)=0 & \textbf{on } \partial\Omega, \end{cases}$$

where

$$\begin{aligned} Lu(x) = &-\mathrm{div}\Big[\omega_1(x)\mathcal{A}(x,\nabla u(x))+\omega_2(x)\mathcal{B}(x,u(x),\nabla u(x))\Big]\\ &+\ \omega_1(x)g(x,u(x))+\ \omega_2(x)\mathcal{H}(x,u(x),\nabla u(x)), \end{aligned}$$

is a second order degenerate elliptic operator, with $\mathcal{A}:\Omega\times\mathbb{R}^n\longrightarrow\mathbb{R}$, $\mathcal{B}:\Omega\times\mathbb{R}\times\mathbb{R}^n\longrightarrow\mathbb{R}$, $g:\Omega\times\mathbb{R}\longrightarrow\mathbb{R}$ and $\mathcal{H}:\Omega\times\mathbb{R}\times\mathbb{R}^n\longrightarrow\mathbb{R}$ are Caratéodory functions, who satisfies some conditions, and the right-hand side term μ belongs to $L^1(\Omega)+\prod_{j=1}^{n}L^{p'}(\Omega,\omega_1^{1-p'})$, ω_1 and ω_2 are weight functions that will be defined in the preliminaries.

***Index Terms*—Nonlinear degenerate elliptic equations, Dirichlet problem, weighted Sobolev spaces, weak solution**

I. Introduction

Let Ω be a bounded open subset in $\mathbb{R}^n$ ($n\geq 2$), $\partial\Omega$ its boundary and $p>1$ and ω_1, ω_2 are two weights functions in Ω (ω_1 and ω_2 are measurable and strictly positive a.e. in Ω). Let us consider the following nonlinear degenerate elliptic problem

$$\begin{cases} Lu(x)=\mu & \text{in } \Omega,\\ u(x)=0 & \text{on } \partial\Omega, \end{cases} \tag{1}$$

where, L is a second order degenerate elliptic operator

$$\begin{aligned} Lu(x) = &-\mathrm{div}\Big[\omega_1(x)\mathcal{A}(x,\nabla u(x))+\omega_2(x)\mathcal{B}(x,u(x),\nabla u(x))\Big]\\ &+\ \omega_1(x)g(x,u(x))+\ \omega_2(x)\mathcal{H}(x,u(x),\nabla u(x)), \end{aligned} \tag{2}$$

and

$$\mu = f_0-\sum_{j=1}^{n}D_jf_j, \tag{3}$$

with $f_0\in L^1(\Omega)$ and for $j=1,...,n,\ f_j\in L^{p'}(\Omega,\omega_1^{1-p'})$. Furthermore, the functions $\mathcal{A}:\Omega\times\mathbb{R}^n\longrightarrow\mathbb{R}$, $\mathcal{B}:\Omega\times\mathbb{R}\times\mathbb{R}^n\longrightarrow\mathbb{R}$, $g:\Omega\times\mathbb{R}\longrightarrow\mathbb{R}$ and $\mathcal{H}:\Omega\times\mathbb{R}\times\mathbb{R}^n\longrightarrow\mathbb{R}$ are Caratéodory functions, who satisfying the assumptions of growth, ellipticity and monotonicity.

In the past decade, much attention has been devoted to nonlinear elliptic equations because of their wide application to physical models such as non-Newtonian fluids, boundary layer phenomena for viscous fluids, and chemical heterogenous model, we mention some works in this direction [1], [4], [5], [7]. One of the motivations for studying (1) comes from applications to electrorheological fluids (see [19] for more details) as an important class of non-Newtonian fluids.

In general, the Sobolev spaces $W^{k,p}(\Omega)$ without weights occur as spaces of solutions for elliptic and parabolic partial difierential equations. For degenerate partial differential equations, where we have equations with various types of singularities in the coefficients, it is natural to look for solutions in weighted Sobolev spaces [2], [4], [12], [13], [16]. The type of a weight depends on the equation type.

For $\omega_1\equiv\omega_2\equiv 1$ (the non weighted case) and $\mathcal{A}(x,\nabla u)\equiv g\equiv 0$, Equation of the from (1) have been widely studied in [10], where the authors obtain some existence results for the solutions (see also the references therein).

Boccardo et al. [6] considered the nonlinear boundary value problem

$$-\mathrm{div}(a(x,u,\nabla u))+g(x,u,\nabla u)=\mu,$$

where $\mu\in L^1(\Omega)+W^{-1,p'}(\Omega)$ and $g(x,u,\nabla u)\in L^1(\Omega)$. By combining the truncation technique with some delicate test functions, the authors showed that the problem has a solution $u\in W_0^{1,p}(\Omega)$. Furthermore the degenerate case with difierent conditions haven been studied by many authors (we refer to [11], [22] for more details).

In [3], the authors proved the existence results, in the setting of weighted Sobolev spaces, for quasilinear degenerated elliptic problems associated with the following equation $-\text{div}\Big(a(x,u,\nabla u)\Big) + g(x,u,\nabla u) = f - \text{div}F$, where g satisfies the sign condition.

In [8] the author proved the existence of solutions for the problem (1) when $\omega_1 \equiv \omega_2$ and $\mathcal{A}(x,\nabla u) \equiv g \equiv 0$. When $\mathcal{H}(x,u,\nabla u) \equiv g \equiv 0$ existence result for the Problem (1) have been shown in [9].

Our objectif, in this paper, is to study equation (1) by adopting Sobolev spaces with weight $W_0^{1,p}(\Omega,\omega_1)$ (see Definition 2.3). By apply the main theorem on monotone operators (see Theorem 2.3), we show that the Problem (1) admits one and only solution $u \in W_0^{1,p}(\Omega,\omega_1)$.

The paper is organized as follows. In Section 2, we give some preliminaries and the definition of weighted Sobolev spaces and some technical lemmas needed in our peper. In Section 3, we make precise all the assumptions on $\mathcal{A}$, $\mathcal{B}$, g, $\mathcal{H}$ and we introduce the notion of weak solution for the Problem (1). Our main result and his proof, the existence and uniqueness of solution to Equation (1), are collected in Section 4. Section 5 is devoted to an example which illustrates our main result.

II. Preliminaries

In this section, we present some definitions, and preliminaries facts which are used throughout this paper.

By a weight, we shall mean a locally integrable function ω on $\mathbb{R}^n$ such that $\omega(x) > 0$ for a.e. $x \in \mathbb{R}^n$. Every weight ω gives rise to a measure on the measurable subsets on $\mathbb{R}^n$ through integration. This measure will also be denoted by ω. Thus,

$$\omega(E) = \int_E \omega(x)dx \qquad \text{for measurable subset} \quad E \subset \mathbb{R}^n.$$

For $0 < p < \infty$, we denote by $L^p(\Omega,\omega)$ the space of measurable functions f on Ω such that

$$||f||_{L^p(\Omega,\omega)} = \left(\int_E |f(x)|^p \omega(x)dx\right)^{\frac{1}{p}} < \infty,$$

where ω is a weight, and Ω be open in $\mathbb{R}^n$.

It is a well-known fact that the space $L^p(\Omega,\omega)$, endowed with this norm is a Banach space. We also have that the dual space of $L^p(\Omega,\omega)$ is the space $L^{p'}(\Omega,\omega^{1-p'})$.

We now determine conditions on the weight ω that guarantee that functions in $L^p(\Omega,\omega)$ are locally integrable on Ω.

Proposition 2.1: [17], [18] Let $1 \le p < \infty$. If the weight ω is such that

$$\omega^{\frac{-1}{p-1}} \in L^1_{loc}(\Omega) \qquad \text{if} \quad p > 1,$$

$$ess \sup_{x\in B} \frac{1}{\omega(x)} < +\infty \qquad \text{if} \quad p = 1,$$

for every ball $B \subset \Omega$. Then,

$$L^p(\Omega,\omega) \subset L^1_{loc}(\Omega).$$

As a consequence, under conditions of Proposition 2.1, the convergence in $L^p(\Omega,\omega)$ implies convergence in $L^1_{loc}(\Omega)$. Moreover, every function in $L^p(\Omega,\omega)$ has a distributional derivatives. It thus makes sense to talk about distributional derivatives of functions in $L^p(\Omega,\omega)$.

A class of weights, which is particularly well understood, is the class of A_p-weight that was introduced by B. Muckenhoupt.

Definition 2.1: Let $1 \le p < \infty$. A weight ω is said to be an A_p-weight, or ω belongs to the Muckenhoupt class, if there exists a positive constant $C = C(p,\omega)$ such that, for every ball $B \subset \mathbb{R}^n$

$$\left(\frac{1}{|B|}\int_B \omega(x)dx\right)\left(\frac{1}{|B|}\int_B (\omega(x))^{\frac{-1}{p-1}}\,dx\right)^{p-1} \le C \text{ if } p > 1,$$

$$\left(\frac{1}{|B|}\int_B \omega(x)dx\right) ess \sup_{x\in B} \frac{1}{\omega(x)} \le C \quad \text{if} \ \ p = 1,$$

where $|.|$ denotes the n-dimensional Lebesgue measure in $\mathbb{R}^n$. The infimum over all such constants C is called the A_p constant of ω. We denote by A_p, $1 \le p < \infty$, the set of all A_p weights.

If $1 \le q \le p < \infty$, then $A_1 \subset A_q \subset A_p$ and the A_q constant of ω equals the A_p constant of ω (we refer to [15], [16], [20] for more informations about A_p-weights).

Example 2.1: (Example of A_p-weights)

(i) If ω is a weight and there exist two positive constants C and D such that $C \le \omega(x) \le D$ for a.e. $x \in \mathbb{R}^n$, then $\omega \in A_p$ for $1 \le p < \infty$.

(ii) Suppose that $\omega(x) = |x|^\eta$, $x \in \mathbb{R}^n$. Then $\omega \in A_p$ if and only if $-n < \eta < n(p-1)$ for $1 \le p < \infty$ (see Corollary 4.4, Chapter IX in [20]).

(iii) Let Ω be an open subset of $\mathbb{R}^n$. Then $\omega(x) = e^{\lambda\varphi(x)} \in A_2$, with $\varphi \in W^{1,n}(\Omega)$ and λ is sufficiently small (see Corollary 2.18 in [15]).

Definition 2.2: A weight ω is said to be doubling, if there exists a positive constant C such that

$$\omega(2B) \le C\omega(B),$$

for every ball $B = B(x,r) \subset \mathbb{R}^n$, where $\omega(B) = \int_B \omega(x)dx$ and $2B$ denotes the ball with the same center as B which is twice as large. The infimum over all constants C is called the doubling constant of ω.

It follows directly from the A_p condition and Hölder inequality that an A_p-weight has the following strong doubling property. In particular, every A_p-weight is doubling (see Corollary 15.7 in [16]).

Proposition 2.2: [21] Let $\omega \in A_p$ with $1 \le p < \infty$ and let E be a measurable subset of a ball $B \subset \mathbb{R}^n$. Then

$$\left(\frac{|E|}{|B|}\right)^p \le C\frac{\omega(E)}{\omega(B)}$$

where C is the A_p constant of ω.

Remark 2.1: If $\omega(E) = 0$ then $|E| = 0$. The measure ω and the Lebesgue measure $|.|$ are mutually absolutely continuous,

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that is they have the same zero sets $\big(\omega(E) = 0$ if and only if $|E| = 0\big)$; so there is no need to specify the measure when using the ubiquitous expression almost everywhere and almost every, both abbreviated a.e..
The weighted Sobolev space $W^{1,p}(\Omega, \omega)$ is defined as follows.

Definition 2.3: Let $\Omega \subset \mathbb{R}^n$ be open, and let ω be an A_p-weight, $1 \leq p < \infty$. We define the weighted Sobolev space $W^{1,p}(\Omega, \omega)$ as the set of functions $u \in L^p(\Omega, \omega)$ with weak derivatives $D_j u \in L^p(\Omega, \omega)$, for $j = 1, ..., n$. The norm of u in $W^{1,p}(\Omega, \omega)$ is given by

$$||u||^p_{W^{1,p}(\Omega,\omega)} = \int_\Omega |u(x)|^p \omega(x) dx + \sum_{j=1}^n \int_\Omega |D_j u(x)|^p \omega(x) dx.$$

We also define $W^{1,p}_0(\Omega, \omega)$ as the closure of $\mathcal{C}^\infty_0(\Omega)$ in $W^{1,p}(\Omega, \omega)$ with respect to the norm $||.||_{W^{1,p}(\Omega,\omega)}$. Note that $C^\infty_0(\Omega)$ is dense in $W^{1,p}_0(\Omega, \omega)$.
Equipped by this norm, $W^{1,p}(\Omega, \omega)$ and $W^{1,p}_0(\Omega, \omega)$ are separable and reflevixe Banach spaces $\big($see Proposition 2.1.2. in [21] and see [18] for more informations about the spaces $W^{1,p}(\Omega, \omega)\big)$. The dual of space $W^{1,p}_0(\Omega, \omega)$ is the space $W^{-1,p'}_0(\Omega, \omega^{1-p'})$.

Let us give the following theorems which are needed later.

Theorem 2.1: [14] Let $\omega \in A_p$, $1 \leq p < \infty$, and let Ω be a bounded open set in $\mathbb{R}^n$. If $u_m \longrightarrow u$ in $L^p(\Omega, \omega)$, then there exist a subsequence (u_{m_k}) and a function $\Phi \in L^p(\Omega, \omega)$ such that

(i) $u_{m_k}(x) \longrightarrow u(x)$, $m_k \longrightarrow \infty$, ω-a.e. on Ω.
(ii) $|u_{m_k}(x)| \leq \Phi(x)$, ω-a.e. on Ω.

Theorem 2.2: [11] (The weighted Sobolev inequality) Let $\omega \in A_p$, $1 \leq p < \infty$, and let Ω be a bounded open set in $\mathbb{R}^n$. There exist constants C_Ω and δ positive such that for all $u \in W^{1,p}_0(\Omega, \omega)$ and all θ satisfying $1 \leq \theta \leq \frac{n}{n-1} + \delta$,

$$||u||_{L^{\theta p}(\Omega,\omega)} \leq C_\Omega ||\nabla u||_{L^p(\Omega,\omega)},$$

where C_Ω depends only on n, p, the A_p constant of ω and the diameter of Ω.

Theorem 2.3: [22] Let $A : X \longrightarrow X^*$ be a monotone, coercive and hemicontinuous operator on the real, separable, reflexive Banach space X. Then the following assertions hold:

(a) For each $T \in X^*$, the equation $Au = T$ has a solution $u \in X$.
(b) If the operator A is strictly monotone, then equation $Au = T$ has a unique solution $u \in X$.

III. Basic assumptions and notion of solutions

A. Basic assumptions

Let us now give the precise hypotheses on the Problem (1), we assume that the following assumptions: Ω be a bounded open subset of $\mathbb{R}^n$($n \geq 2$), $1 < q < p < \infty$, let ω_1 and ω_2 are two weights functions, and let $\mathcal{A}_j : \Omega \times \mathbb{R}^n \longrightarrow \mathbb{R}$, $\mathcal{B}_j : \Omega \times \mathbb{R} \times \mathbb{R}^n \longrightarrow \mathbb{R}$ $(j = 1, ..., n)$, with $\mathcal{B}(x, \eta, \xi) = \Big(\mathcal{B}_1(x, \eta, \xi), ..., \mathcal{B}_n(x, \eta, \xi)\Big)$ and $\mathcal{A}(x, \xi) = \Big(\mathcal{A}_1(x, \xi), ..., \mathcal{A}_n(x, \xi)\Big)$, $g : \Omega \times \mathbb{R} \longrightarrow \mathbb{R}$ and $\mathcal{H} : \Omega \times \mathbb{R} \times \mathbb{R}^n \longrightarrow \mathbb{R}$ satisfying the following assumptions:

(A1) For $j = 1, ..., n$, $\mathcal{B}_j$, $\mathcal{A}_j$, g and $\mathcal{H}$ are Caratéodory functions.

(A2) There are positive functions $h_1, h_2, h_3, h_4, h_5, h_6 \in L^\infty(\Omega)$ and $K_1, K_4 \in L^{p'}(\Omega, \omega_1)\Big($with $\frac{1}{p} + \frac{1}{p'} = 1 \Big)$ and $K_2, K_3 \in L^{q'}(\Omega, \omega_2)\Big($with $\frac{1}{q} + \frac{1}{q'} = 1\Big)$ such that :

$$|\mathcal{A}(x, \xi)| \leq K_1(x) + h_1(x)|\xi|^{\frac{p}{p'}},$$

$$|\mathcal{B}(x, \eta, \xi)| \leq K_2(x) + h_2(x)|\eta|^{\frac{q}{q'}} + h_3(x)|\xi|^{\frac{q}{q'}},$$

$$|g(x, \eta)| \leq K_4(x) + h_6(x)|\eta|^{\frac{p}{p'}},$$

and

$$|\mathcal{H}(x, \eta, \xi)| \leq K_3(x) + h_4(x)|\eta|^{\frac{q}{q'}} + h_5(x)|\xi|^{\frac{q}{q'}}.$$

(A3) There exists a constant $\alpha > 0$ such that :

$$\langle \mathcal{A}(x, \xi) - \mathcal{A}(x, \xi'), \xi - \xi' \rangle \geq \alpha|\xi - \xi'|^p,$$

$$\langle \mathcal{B}(x, \eta, \xi) - \mathcal{B}(x, \eta', \xi'), \xi - \xi' \rangle \geq 0,$$

$$\Big(g(x, \eta) - g(x, \eta')\Big)\Big(\eta - \eta'\Big) \geq 0,$$

and

$$\Big(\mathcal{H}(x, \eta, \xi) - \mathcal{H}(x, \eta', \xi')\Big)\Big(\eta - \eta'\Big) \geq 0,$$

whenever $(\eta, \xi), (\eta', \xi') \in \mathbb{R} \times \mathbb{R}^n$ with $\eta \neq \eta'$ and $\xi \neq \xi'$ $\Big($where $\langle ., . \rangle$ denotes here the usual inner product in $\mathbb{R}^n\Big)$.

(A4) There are constants $\lambda_1, \lambda_2, \lambda_3, \lambda_4 > 0$ such that :

$$\langle \mathcal{A}(x, \xi), \xi \rangle \geq \lambda_1 |\xi|^p,$$

$$\langle \mathcal{B}(x, \eta, \xi), \xi \rangle \geq \lambda_2 |\xi|^q + \lambda_3 |\eta|^q,$$

$$g(x, \eta)\eta \geq \lambda_4 |\eta|^p,$$

and

$$\mathcal{H}(x, \eta, \xi)\eta \geq 0.$$

B. Notions of solutions

The definition of a weak solution for Problem (1) can be stated as follows.

Definition 3.1: We say that an element $u \in W^{1,p}_0(\Omega, \omega_1)$ is a weak solution of Problem (1) if :

$$\int_\Omega \langle \mathcal{A}(x, \nabla u), \nabla \varphi \rangle \omega_1 dx + \int_\Omega \langle \mathcal{B}(x, u, \nabla u), \nabla \varphi \rangle \omega_2 dx + \int_\Omega g(x, u)\varphi \omega_1 dx + \int_\Omega \mathcal{H}(x, u, \nabla u)\varphi \omega_2 dx = \int_\Omega f_0 \varphi dx + \sum_{j=1}^n \int_\Omega f_j D_j \varphi dx,$$

for all $\varphi \in W^{1,p}_0(\Omega, \omega_1)$.

Remark 3.1: We seek to establish a relationship between ω_1 and ω_2, in order to ensure the existence and uniqueness of solution for our Problem (1). At first we notice if $\frac{\omega_2}{\omega_1} \in$

$L^s(\Omega,\omega_1)$ where $s=\frac{p}{p-q}$, $1<q<p<\infty$ and $\omega_1,\omega_2\in A_p$, then, by Hölder inequality we obtain

$$||u||_{L^q(\Omega,\omega_2)}\leq C_{p,q}||u||_{L^p(\Omega,\omega_1)},$$

where $C_{p,q}=||\frac{\omega_2}{\omega_1}||^{1/q}_{L^s(\Omega,\omega_1)}$.

IV. Main result

A. Result on the existence and uniqueness

In this subsection we will state the existence and uniqueness of solution to Problem (1) in Theorem 4.1. In the next subsections we will present the proof.

Theorem 4.1: Let $1<q<p<\infty$ and assume that **(A1)** – **(A4)** holds. If

(i) $f_0/\omega_2\in L^{q'}(\Omega,\omega_2)$ and $f_j/\omega_1\in L^{p'}(\Omega,\omega_1)$ $(j=1,...,n)$..

(ii) $\omega_1,\omega_2\in A_p$ such that $\frac{\omega_2}{\omega_1}\in L^s(\Omega,\omega_1)$, where $s=\frac{p}{p-q}$.

Then, the Problem (1) has only one solution $u\in W^{1,p}_0(\Omega,\omega_1)$.

B. Proof of Theorem 4.1

The basic idea of our proof is to reduce the Problem (1) to an operator equation $\mathbf{A}u=\mathbf{T}$ and apply the Theorem 2.3. The proof of Theorem 4.1 will be divided into several steps.

1) Equivalent operator equation: In this subsection, we use the somme tools and the condition **(A2)** to prove an existence the operator $\mathbf{A}$ such that the Problem (1) is equivalent to the operator equation $\mathbf{A}u=\mathbf{T}$. We introduce the operators

$$\mathbf{T}: W^{1,p}_0(\Omega,\omega_1)\longrightarrow\mathbb{R}$$
$$\varphi\longrightarrow\mathbf{T}(\varphi)=\int_\Omega f_0\varphi dx+\sum_{j=1}^n\int_\Omega f_jD_j\varphi dx,$$

and

$$\mathbf{B}: W^{1,p}_0(\Omega,\omega_1)\times W^{1,p}_0(\Omega,\omega_1)\longrightarrow\mathbb{R}$$
$$(u,\varphi)\longrightarrow\mathbf{B}_1(u,\varphi)+\mathbf{B}_2(u,\varphi)+\mathbf{B}_3(u,\varphi)+\mathbf{B}_4(u,\varphi),$$

where, $\mathbf{B}_1$, $\mathbf{B}_2$, $\mathbf{B}_3$ and $\mathbf{B}_4$ are defined as follows

$$\mathbf{B}_1: W^{1,p}_0(\Omega,\omega_1)\times W^{1,p}_0(\Omega,\omega_1)\longrightarrow\mathbb{R}$$
$$\mathbf{B}_1(u,\varphi)=\int_\Omega\langle\mathcal{A}(x,\nabla u),\nabla\varphi\rangle\omega_1dx,$$

$$\mathbf{B}_2: W^{1,p}_0(\Omega,\omega_1)\times W^{1,p}_0(\Omega,\omega_1)\longrightarrow\mathbb{R}$$
$$\mathbf{B}_2(u,\varphi)=\int_\Omega\langle\mathcal{B}(x,u,\nabla u),\nabla\varphi\rangle\omega_2dx,$$

$$\mathbf{B}_3: W^{1,p}_0(\Omega,\omega_1)\times W^{1,p}_0(\Omega,\omega_1)\longrightarrow\mathbb{R}$$
$$\mathbf{B}_3(u,\varphi)=\int_\Omega g(x,u)\varphi\omega_1dx.$$

$$\mathbf{B}_4: W^{1,p}_0(\Omega,\omega_1)\times W^{1,p}_0(\Omega,\omega_1)\longrightarrow\mathbb{R}$$
$$\mathbf{B}_4(u,\varphi)=\int_\Omega\mathcal{H}(x,u,\nabla u)\varphi\omega_2dx.$$

Then $u\in W^{1,p}_0(\Omega,\omega_1)$ is a weak solution of Problem (1) if and only if

$$\mathbf{B}(u,\varphi)=\mathbf{T}(\varphi),\qquad\text{for all }\ \varphi\in W^{1,p}_0(\Omega,\omega_1).$$

We will show that $\mathbf{T}\in W^{-1,p'}_0(\Omega,\omega_1^{1-p'})$ and $\mathbf{B}(u,.)$ is linear, for each $u\in W^{1,p}_0(\Omega,\omega_1)$.

(i) Using Hölder inequality and Theorem 2.2(with $\theta=1$), we obtain

$$\begin{aligned}&|\mathbf{T}(\varphi)|\\&\leq\int_\Omega|f_0|\,|\varphi|\,dx+\sum_{j=1}^n\int_\Omega|f_j|\,|D_j\varphi|\,dx\\&\leq\left(C_{p,q}||f_0/\omega_2||_{L^{q'}(\Omega,\omega_2)}+\sum_{j=1}^n||f_j/\omega_1||_{L^{p'}(\Omega,\omega_1)}\right)||\varphi||_{W^{1,p}_0(\Omega,\omega_1)}.\end{aligned}$$

According to $f_0/\omega_2\in L^{q'}(\Omega,\omega_2)$ and $f/\omega_1\in L^{p'}(\Omega,\omega_1)$, we deduce that $\mathbf{T}\in W^{-1,p'}_0(\Omega,\omega_1^*)$.

(ii) Let $u\in W^{1,p}_0(\Omega,\omega_1)$. We have

$$|\mathbf{B}(u,\varphi)|\ \leq\ |\mathbf{B}_1(u,\varphi)|+|\mathbf{B}_2(u,\varphi)|+|\mathbf{B}_3(u,\varphi)|+|\mathbf{B}_4(u,\varphi)|.\tag{4}$$

In (4), by **(A2)**, Hölder inequality, Remark 3.1 and Theorem 2.2(with $\theta=1$), we have

$$\begin{aligned}|\mathbf{B}_1(u,\varphi)|&\leq\int_\Omega|\mathcal{A}(x,\nabla u)||\nabla\varphi|\omega_1dx\\&\leq\int_\Omega\left(K_1+h_1|\nabla u|^{\frac{p}{p'}}\right)|\nabla\varphi|\omega_1dx\\&\leq\left(||K_1||_{L^{p'}(\Omega,\omega_1)}+||h_1||_{L^\infty(\Omega)}||u||^{\frac{p}{p'}}_{W^{1,p}_0(\Omega,\omega_1)}\right)||\varphi||_{W^{1,p}_0(\Omega,\omega_1)},\end{aligned}$$

and

$$\begin{aligned}|\mathbf{B}_2(u,\varphi)|&\leq\int_\Omega|\mathcal{B}(x,u,\nabla u)||\nabla\varphi|\omega_2dx\\&\leq\int_\Omega\left(K_2+h_2|u|^{\frac{q}{q'}}+h_3|\nabla u|^{\frac{q}{q'}}\right)|\nabla\varphi|\omega_2dx\\&\leq||K_2||_{L^{q'}(\Omega,\omega_2)}C_{p,q}||\nabla\varphi||_{L^p(\Omega,\omega_1)}+||h_2||_{L^\infty(\Omega)}C^{\frac{q}{q'}}_{p,q}||u||^{\frac{q}{q'}}_{L^p(\Omega,\omega_1)}\\&\quad C_{p,q}||\nabla\varphi||_{L^p(\Omega,\omega_1)}+||h_3||_{L^\infty(\Omega)}C^{\frac{q}{q'}}_{p,q}||\nabla u||^{\frac{q}{q'}}_{L^p(\Omega,\omega_1)}C_{p,q}||\nabla\varphi||_{L^p(\Omega,\omega_1)}\\&\leq\Big[C_{p,q}||K_2||_{L^{q'}(\Omega,\omega_2)}+C^q_{p,q}\left(||h_2||_{L^\infty(\Omega)}+||h_3||_{L^\infty(\Omega)}\right)\\&\qquad||u||^{q-1}_{W^{1,p}_0(\Omega,\omega_1)}\Big]||\varphi||_{W^{1,p}_0(\Omega,\omega_1)}.\end{aligned}$$

Analogously, we have

$$\begin{aligned}|\mathbf{B}_3(u,\varphi)|&\leq\int_\Omega|g(x,u)||\varphi|\omega_1dx\\&\leq\left(||K_4||_{L^{p'}(\Omega,\omega_1)}+||h_6||_{L^\infty(\Omega)}||u||^{\frac{p}{p'}}_{W^{1,p}_0(\Omega,\omega_1)}\right)||\varphi||_{W^{1,p}_0(\Omega,\omega_1)},\end{aligned}$$

and

$$\begin{aligned}|\mathbf{B}_4(u,\varphi)|&\leq\int_\Omega|\mathcal{H}(x,u,\nabla u)||\varphi|\omega_2dx\\&\leq\Big[C_{p,q}||K_3||_{L^{q'}(\Omega,\omega_2)}+C^q_{p,q}\left(||h_4||_{L^\infty(\Omega)}+||h_5||_{L^\infty(\Omega)}\right)\\&\qquad||u||^{q-1}_{W^{1,p}_0(\Omega,\omega_1)}\Big]||\varphi||_{W^{1,p}_0(\Omega,\omega_1)}.\end{aligned}$$

Hence, in (4) we obtain, for all $u \in W_0^{1,p}(\Omega,\omega_1)$,

$$\begin{aligned}&|\mathbf{B}(u,\varphi)|\\ \leq &\Big[\|K_1\|_{L^{p'}(\Omega,\omega_1)} + \|K_4\|_{L^{p'}(\Omega,\omega_1)} + C_{p,q}\Big(\|K_3\|_{L^{q'}(\Omega,\omega_2)}\\ &+\|K_2\|_{L^{q'}(\Omega,\omega_2)}\Big) + \Big(\|h_1\|_{L^\infty(\Omega)} + \|h_6\|_{L^\infty(\Omega)}\Big)\|u\|^{\frac{p}{p'}}_{W_0^{1,p}(\Omega,\omega_1)}\\ &+C^q_{p,q}\Big(\|h_2\|_{L^\infty(\Omega)} + \|h_3\|_{L^\infty(\Omega)} + \|h_4\|_{L^\infty(\Omega)} + \|h_5\|_{L^\infty(\Omega)}\Big)\\ &\|u\|^{q-1}_{W_0^{1,p}(\Omega,\omega_1)}\Big]\|\varphi\|_{W_0^{1,p}(\Omega,\omega_1)}.\end{aligned}$$

Since $\mathbf{B}(u,.)$ is linear and continuous, for each $u \in W_0^{1,p}(\Omega,\omega_1)$, there exists a linear and continuous operator denoted by $\mathbf{A} : W_0^{1,p}(\Omega,\omega_1) \longrightarrow W_0^{-1,p'}(\Omega,\omega_1^{1-p'})$ such that

$$\langle \mathbf{A}u,\varphi\rangle = \mathbf{B}(u,\varphi), \quad \text{for all } u,\varphi \in W_0^{1,p}(\Omega,\omega_1),$$

where $\langle f,x\rangle$ denotes the value of the linear functional f at the point x . Moreover, we have

$$\begin{aligned}&\|\mathbf{A}u\|_*\\ \leq &\|K_1\|_{L^{p'}(\Omega,\omega_1)} + \|K_4\|_{L^{p'}(\Omega,\omega_1)} + C_{p,q}\Big(\|K_3\|_{L^{q'}(\Omega,\omega_2)}\\ &+\|K_2\|_{L^{q'}(\Omega,\omega_2)}\Big) + \Big(\|h_1\|_{L^\infty(\Omega)} + \|h_6\|_{L^\infty(\Omega)}\Big)\|u\|^{\frac{p}{p'}}_{W_0^{1,p}(\Omega,\omega_1)}\\ &+C^q_{p,q}\Big(\|h_2\|_{L^\infty(\Omega)} + \|h_3\|_{L^\infty(\Omega)} + \|h_4\|_{L^\infty(\Omega)} + \|h_5\|_{L^\infty(\Omega)}\Big)\\ &\|u\|^{q-1}_{W_0^{1,p}(\Omega,\omega_1)},\end{aligned}$$

where

$$\|\mathbf{A}u\|_* = \sup\Big\{|\langle \mathbf{A}u,\varphi\rangle| = |\mathbf{B}(u,\varphi)| : \|\varphi\|_{W_0^{1,p}(\Omega,\omega_1)} = 1\Big\}$$

is the norm in $W_0^{-1,p'}(\Omega,\omega_1^{1-p'})$.

Consequently, Problem (1) is equivalent to the operator equation

$$\mathbf{A}u = \mathbf{T},\ u \in W_0^{1,p}(\Omega,\omega_1).$$

2) Coercivity of the operator $\mathbf{A}$: In this step, we prove that the operator $\mathbf{A}$ is coercive. To this purpose let $u \in W_0^{1,p}(\Omega,\omega_1)$, we have

$$\begin{aligned}\langle \mathbf{A}u,u\rangle &= \mathbf{B}(u,u)\\ &= \int_\Omega \langle \mathcal{A}(x,\nabla u),\nabla u\rangle\omega_1 dx + \int_\Omega \langle \mathcal{B}(x,u,\nabla u),\nabla u\rangle\omega_2 dx\\ &\quad+ \int_\Omega g(x,u)u\omega_1 dx + \int_\Omega \mathcal{H}(x,u,\nabla u)u\omega_2 dx.\end{aligned}$$

Moreover, from **(A4)** and Theorem 2.2(with $\theta = 1$), we obtain

$$\begin{aligned}\langle \mathbf{A}u,u\rangle &\geq \lambda_1\int_\Omega |\nabla u|^p\omega_1 dx + \lambda_2\int_\Omega |\nabla u|^q\omega_2 dx\\ &\quad+\lambda_3\int_\Omega |u|^q\omega_2 dx + \lambda_4\int_\Omega |u|^p\omega_1 dx\\ &\geq min(\lambda_1,\lambda_4)\Big[\int_\Omega |\nabla u|^p\omega_1 dx + \int_\Omega |u|^p\omega_1 dx\Big]\\ &\quad+min(\lambda_2,\lambda_3)\Big[\int_\Omega |\nabla u|^q\omega_2 dx + \int_\Omega |u|^q\omega_2 dx\Big]\\ &= min(\lambda_1,\lambda_4)\|u\|^p_{W_0^{1,p}(\Omega,\omega_1)} + min(\lambda_2,\lambda_3)\|u\|^q_{W_0^{1,q}(\Omega,\omega_2)}\\ &\geq min(\lambda_1,\lambda_4)\|u\|^p_{W_0^{1,p}(\Omega,\omega_1)}.\end{aligned}$$

Hence, we obtain

$$\frac{\langle \mathbf{A}u,u\rangle}{\|u\|_{W_0^{1,p}(\Omega,\omega_1)}} \geq min(\lambda_1,\lambda_4)\|u\|^{p-1}_{W_0^{1,p}(\Omega,\omega_1)}.$$

Therefore, since $p > 1$, we have

$$\frac{\langle \mathbf{A}u,u\rangle}{\|u\|_{W_0^{1,p}(\Omega,\omega_1)}} \longrightarrow +\infty \text{ as } \|u\|_{W_0^{1,p}(\Omega,\omega_1)} \longrightarrow +\infty,$$

that is, $\mathbf{A}$ is coercive.

3) Monotonicity of the operator $\mathbf{A}$: The operator $\mathbf{A}$ is strictly monotone. In fact, for all $u_1,u_2 \in W_0^{1,p}(\Omega,\omega_1)$ with $u_1 \neq u_2$, we have

$$\begin{aligned}&\langle \mathbf{A}u_1 - \mathbf{A}u_2, u_1-u_2\rangle = \mathbf{B}(u_1,u_1-u_2) - \mathbf{B}(u_2,u_1-u_2)\\ &= \int_\Omega \langle \mathcal{A}(x,\nabla u_1) - \mathcal{A}(x,\nabla u_2),\nabla(u_1-u_2)\rangle\omega_1 dx\\ &+ \int_\Omega \langle \mathcal{B}(x,u_1,\nabla u_1) - \mathcal{B}(x,u_2,\nabla u_2),\nabla(u_1-u_2)\rangle\omega_2 dx\\ &+ \int_\Omega \Big(g(x,u_1) - g(x,u_2)\Big)\Big(u_1-u_2\Big)\omega_1 dx\\ &+ \int_\Omega \Big(\mathcal{H}(x,u_1,\nabla u_1) - \mathcal{H}(x,u_2,\nabla u_2)\Big)\Big(u_1-u_2\Big)\omega_2 dx.\end{aligned}$$

Thanks to **(A3)**, we obtain

$$\begin{aligned}\langle \mathbf{A}u_1 - \mathbf{A}u_2, u_1-u_2\rangle &\geq \int_\Omega \alpha|\nabla(u_1-u_2)|^p\omega_1 dx\\ &\geq \alpha\|\nabla(u_1-u_2)\|^p_{L^p(\Omega,\omega_1)},\end{aligned}$$

and by Theorem 2.2(with $\theta = 1$), we conclude that

$$\langle \mathbf{A}u_1 - \mathbf{A}u_2, u_1-u_2\rangle \geq \frac{\alpha}{(C^p_\Omega + 1)}\|u_1-u_2\|^p_{W_0^{1,p}(\Omega,\omega_1)}.$$

Therefore, the operator $\mathbf{A}$ is strictly monotone.

4) Continuity of the operator $\mathbf{A}$: We need to show that the operator $\mathbf{A}$ is continuous. To this purpose let $u_m \longrightarrow u$ in $W_0^{1,p}(\Omega,\omega_1)$ as $m \longrightarrow \infty$. Note that if $u_m \longrightarrow u$ in $W_0^{1,p}(\Omega,\omega_1)$, then $u_m \longrightarrow u$ in $L^p(\Omega,\omega_1)$ et $\nabla u_m \longrightarrow \nabla u$ in $(L^p(\Omega,\omega_1))^n$. Hence, thanks to Theorem 2.1, there exist a subsequence (u_{m_k}), functions $\Phi_1 \in L^p(\Omega,\omega_1)$ and $\Phi_2 \in L^p(\Omega,\omega_1)$ such that

$$\begin{aligned}&u_{m_k}(x) \longrightarrow u(x), && \omega_1 - a.e. \text{ in } \Omega\\ &|u_{m_k}(x)| \leq \Phi_1(x), && \omega_1 - a.e. \text{ in } \Omega\\ &\nabla u_{m_k}(x) \longrightarrow \nabla u(x), && \omega_1 - a.e. \text{ in } \Omega\\ &|\nabla u_{m_k}(x)| \leq \Phi_2(x), && \omega_1 - a.e. \text{ in } \Omega.\end{aligned} \quad (5)$$

We will show that $\mathbf{A}u_m \longrightarrow \mathbf{A}u$ in $W_0^{-1,p'}(\Omega,\omega_1^{1-p'})$. In order to prove this convergence we proceed in four steps.

Step 1:

For $j = 1,...,n$, we define the operator

$$\begin{aligned}&F_j : W_0^{1,p}(\Omega,\omega_1) \longrightarrow L^{p'}(\Omega,\omega_1)\\ &(F_j u)(x) = \mathcal{A}_j(x,\nabla u(x)).\end{aligned}$$

We now show that the operator F_j is bounded and continuous.

(i) Let $u \in W_0^{1,p}(\Omega,\omega_1)$. Using **(A2)** and Theorem 2.2(with $\theta = 1$), we obtain

$$\begin{aligned}\|F_j u\|^{p'}_{L^{p'}(\Omega,\omega_1)} &= \int_\Omega |\mathcal{A}_j(x,\nabla u)|^{p'}\omega_1 dx\\ &\le \int_\Omega \left(K_1 + h_1|\nabla u|^{\frac{p}{p'}}\right)^{p'}\omega_1 dx\\ &\le C_p \int_\Omega \left(K_1^{p'} + h_1^{p'}|\nabla u|^p\right)\omega_1 dx\\ &\le C_p\left[\|K_1\|^{p'}_{L^{p'}(\Omega,\omega_1)} + \|h_1\|^{p'}_{L^\infty(\Omega)}\|\nabla u\|^p_{L^p(\Omega,\omega_1)}\right]\\ &\le C_p\left[\|K_1\|^{p'}_{L^{p'}(\Omega,\omega_1)} + \|h_1\|^{p'}_{L^\infty(\Omega)}\|u\|^p_{W_0^{1,p}(\Omega,\omega_1)}\right],\end{aligned}$$

where the constant C_p depends only on p.

(ii) Let $u_m \longrightarrow u$ in $W_0^{1,p}(\Omega,\omega_1)$ as $m \longrightarrow \infty$. We need to show that $F_j u_m \longrightarrow F_j u$ in $L^{p'}(\Omega,\omega_1)$. We will apply the Lebesgue's theorem and the convergence principle in Banach spaces.
By **(A2)**, we obtain

$$\begin{aligned}&\|F_j u_{m_k} - F_j u\|^{p'}_{L^{p'}(\Omega,\omega_1)}\\ &= \int_\Omega |F_j u_{m_k}(x) - F_j u(x)|^{p'}\omega_1 dx\\ &\le \int_\Omega \left(|\mathcal{A}_j(x,\nabla u_{m_k})| + |\mathcal{A}_j(x,\nabla u)|\right)^{p'}\omega_1 dx\\ &\le C_p\int_\Omega \left(|\mathcal{A}_j(x,\nabla u_{m_k})|^{p'} + |\mathcal{A}_j(x,\nabla u)|^{p'}\right)\omega_1 dx\\ &\le C_p\int_\Omega \left[\left(K_1 + h_1|\nabla u_{m_k}|^{\frac{p}{p'}}\right)^{p'} + \left(K_1 + h_1|\nabla u|^{\frac{p}{p'}}\right)^{p'}\right]\omega_1 dx\\ &\le 2C_pC_p'\int_\Omega \left(K_1^{p'} + h_1^{p'}\Phi_2^p\right)\omega_1 dx\\ &\le 2C_pC_p'\left[\|K_1\|^{p'}_{L^{p'}(\Omega,\omega_1)} + \|h_1\|^{p'}_{L^\infty(\Omega)}\|\Phi_2\|^p_{L^p(\Omega,\omega_1)}\right].\end{aligned}$$

Hence, thanks to **(A1)**, we get, as $k \longrightarrow \infty$

$$F_j u_{m_k}(x) = \mathcal{A}_j(x,\nabla u_{m_k}(x)) \longrightarrow \mathcal{A}_j(x,\nabla u(x)) = F_j u(x),$$

for almost all $x \in \Omega$. Therefore, by Lebesgue's theorem, we obtain

$$\|F_j u_{m_k} - F_j u\|_{L^{p'}(\Omega,\omega_1)} \longrightarrow 0,$$

that is,

$$F_j u_{m_k} \longrightarrow F_j u \quad \text{in} \quad L^{p'}(\Omega,\omega_1).$$

Finally, in view to convergence principle in Banach spaces, we have

$$F_j u_m \longrightarrow F_j u \quad \text{in} \quad L^{p'}(\Omega,\omega_1). \tag{6}$$

Step 2:
For $j = 1,...,n$, we define the operator

$$\begin{aligned}&G_j : W_0^{1,p}(\Omega,\omega_1) \longrightarrow L^{q'}(\Omega,\omega_2)\\ &(G_j u)(x) = \mathcal{B}_j(x,u(x),\nabla u(x)).\end{aligned}$$

We also have that the operator G_j is continuous and bounded. In fact,

(i) Let $u \in W_0^{1,p}(\Omega,\omega_1)$. Using **(A2)**, Remark 3.1 and Theorem 2.2(with $\theta = 1$), we obtain

$$\begin{aligned}\|G_j u\|^{q'}_{L^{q'}(\Omega,\omega_2)} &= \int_\Omega |\mathcal{B}_j(x,u,\nabla u)|^{q'}\omega_2 dx\\ &\le \int_\Omega \left(K_2 + h_2|u|^{\frac{q}{q'}} + h_3|\nabla u|^{\frac{q}{q'}}\right)^{q'}\omega_2 dx\\ &\le C_q\int_\Omega \left[K_2^{q'} + h_2^{q'}|u|^q + h_3^{q'}|\nabla u|^q\right]\omega_2 dx\\ &\le C_q\Big[\|K_2\|^{q'}_{L^{q'}(\Omega,\omega_2)} + \|h_2\|^{q'}_{L^\infty(\Omega)}\|u\|^q_{L^q(\Omega,\omega_2)}\\ &\quad + \|h_3\|^{q'}_{L^\infty(\Omega)}\|\nabla u\|^q_{L^q(\Omega,\omega_2)}\Big]\\ &\le C_q\Big[\|K_2\|^{q'}_{L^{q'}(\Omega,\omega_2)} + \|h_2\|^{q'}_{L^\infty(\Omega)}C^q_{p,q}\|u\|^q_{L^p(\Omega,\omega_1)}\\ &\quad + \|h_3\|^{q'}_{L^\infty(\Omega)}C^q_{p,q}\|\nabla u\|^q_{L^p(\Omega,\omega_1)}\Big]\\ &\le C_q\Big[\|K_2\|^{q'}_{L^{q'}(\Omega,\omega_2)} + C^q_{p,q}\Big(\|h_2\|^{q'}_{L^\infty(\Omega)}\\ &\quad + \|h_3\|^{q'}_{L^\infty(\Omega)}\Big)\|u\|^q_{W_0^{1,p}(\Omega,\omega_1)}\Big],\end{aligned}$$

where the constant C_q depends only on q.

(ii) Let $u_m \longrightarrow u$ in $W_0^{1,p}(\Omega,\omega_1)$ as $m \longrightarrow \infty$. We will show that $G_j u_m \longrightarrow G_j u$ in $L^{q'}(\Omega,\omega_2)$.
According to **(A2)** and Remark 3.1, we obtain

$$\begin{aligned}&\|G_j u_{m_k} - G_j u\|^{q'}_{L^{q'}(\Omega,\omega_2)} = \int_\Omega |G_j u_{m_k}(x) - G_j u(x)|^{q'}\omega_2 dx\\ &\le \int_\Omega \Big(|\mathcal{B}_j(x,u_{m_k},\nabla u_{m_k}| + |\mathcal{B}_j(x,u,\nabla u)|\Big)^{q'}\omega_2 dx\\ &\le C_q\int_\Omega \left(|\mathcal{B}_j(x,u_{m_k},\nabla u_{m_k})|^{q'} + |\mathcal{B}_j(x,u,\nabla u)|^{q'}\right)\omega_2 dx\\ &\le C_q\Big[\int_\Omega \left(K_2 + h_2|u_{m_k}|^{\frac{q}{q'}} + h_3|\nabla u_{m_k}|^{\frac{q}{q'}}\right)^{q'}\omega_2 dx\\ &\quad + \int_\Omega \left(K_2 + h_2|u|^{\frac{q}{q'}} + h_3|\nabla u|^{\frac{q}{q'}}\right)^{q'}\omega_2 dx\Big]\\ &\le 2C_qC_q'\int_\Omega \left(K_2^{q'} + h_2^{q'}\Phi_1^q + h_3^{q'}\Phi_2^q\right)\omega_2 dx\\ &\le 2C_qC_q'\Big[\|K_2\|^{q'}_{L^{q'}(\Omega,\omega_2)} + \|h_2\|^{q'}_{L^\infty(\Omega)}\|\Phi_1\|^q_{L^q(\Omega,\omega_2)}\\ &\quad + \|h_3\|^{q'}_{L^\infty(\Omega)}\|\Phi_2\|^q_{L^q(\Omega,\omega_2)}\Big]\\ &\le 2C_qC_q'\Big[\|K_2\|^{q'}_{L^{q'}(\Omega,\omega_2)} + C^q_{p,q}\|h_2\|^{q'}_{L^\infty(\Omega)}\|\Phi_1\|^q_{L^p(\Omega,\omega_1)}\\ &\quad + C^q_{p,q}\|h_3\|^{q'}_{L^\infty(\Omega)}\|\Phi_2\|^q_{L^p(\Omega,\omega_1)}\Big].\end{aligned}$$

Then, by **(A1)**, we have, as $k \longrightarrow \infty$

$$G_j u_{m_k}(x) \longrightarrow G_j u(x), \quad \text{a.e. } x \in \Omega.$$

Therefore, in view to Lebesgue's theorem, we have

$$\|G_j u_{m_k} - G_j u\|_{L^{q'}(\Omega,\omega_2)} \longrightarrow 0,$$

that is,

$$G_j u_{m_k} \longrightarrow G_j u \quad \text{in} \quad L^{q'}(\Omega,\omega_2).$$

Hence, from the convergence principle in Banach spaces, we conclude that

$$G_j u_m \longrightarrow G_j u \quad \text{in} \quad L^{q'}(\Omega,\omega_2). \tag{7}$$

Step 3:
We define the operator

$$H : W_0^{1,p}(\Omega,\omega_1) \longrightarrow L^{p'}(\Omega,\omega_1)$$
$$(Hu)(x) = g(x,u(x)).$$

In this step, we will show that the operator H is bounded and continuous.

(i) Let $u \in W_0^{1,p}(\Omega,\omega_1)$. Using **(A2)**, we obtain

$$\begin{aligned}\|Hu\|^{p'}_{L^{p'}(\Omega,\omega_1)} &= \int_\Omega |g(x,u)|^{p'}\omega_1 dx \\ &\le \int_\Omega \left(K_4 + h_6|u|^{\frac{p}{p'}}\right)^{p'}\omega_1 dx \\ &\le C_p \int_\Omega \left(K_4^{p'} + h_6^{p'}|u|^p\right)\omega_1 dx \\ &\le C_p\left[\|K_4\|^{p'}_{L^{p'}(\Omega,\omega_1)} + \|h_6\|^{p'}_{L^\infty(\Omega)}\|u\|^p_{L^p(\Omega,\omega_1)}\right] \\ &\le C_p\left[\|K_3\|_{L^{p'}(\Omega,\omega_1)} + \|h_6\|^{p'}_{L^\infty(\Omega)}\|u\|^p_{W_0^{1,p}(\Omega,\omega_1)}\right],\end{aligned}$$

where the constant C_p depends only on p.

(ii) Let $u_m \longrightarrow u$ in $W_0^{1,p}(\Omega,\omega_1)$ as $m \longrightarrow \infty$. We need to show that $Hu_m \longrightarrow Hu$ in $L^{p'}(\Omega,\omega_1)$.
By **(A2)** , we get

$$\begin{aligned}&\|Hu_{m_k} - Hu\|^{p'}_{L^{p'}(\Omega,\omega_1)} \\ &= \int_\Omega |Hu_{m_k}(x) - hu(x)|^{p'}\omega_1 dx \\ &\le \int_\Omega (|g(x,u_{m_k})| + |g(x,u)|)^{p'}\omega_1 dx \\ &\le C_p \int_\Omega \left(|g(x,u_{m_k})|^{p'} + |g(x,u)|^{p'}\right)\omega_1 dx \\ &\le C_p \int_\Omega \left[(K_4 + h_6|u_{m_k}|^{\frac{p}{p'}})^{p'} + (K_4 + h_6|u|^{\frac{p}{p'}})^{p'}\right]\omega_1 dx \\ &\le 2C_pC'_p \int_\Omega \left(K_4^{p'} + h_6^{p'}\Phi_1^p\right)\omega_1 dx \\ &\le 2C_pC'_p\left[\|K_4\|^{p'}_{L^{p'}(\Omega,\omega_1)} + \|h_6\|^{p'}_{L^\infty(\Omega)}\|\Phi_1\|^p_{L^p(\Omega,\omega_1)}\right],\end{aligned}$$

then, using condition **(H1)**, we deduce, as $k \longrightarrow \infty$

$$Hu_{m_k}(x)) \longrightarrow Hu(x), \quad \text{a.e. } x \in \Omega.$$

Therefore, by the Lebesgue's theorem, we obtain

$$\|Hu_{m_k} - Hu\|_{L^{p'}(\Omega,\omega_1)} \longrightarrow 0,$$

that is,

$$Hu_{m_k} \longrightarrow Hu \quad \text{in} \quad L^{p'}(\Omega,\omega_1).$$

We conclude, from the convergence principle in Banach spaces, that

$$Hu_m \longrightarrow Hu \quad \text{in} \quad L^{p'}(\Omega,\omega_1). \tag{8}$$

Step 4:
We define the operator

$$\tilde{H} : W_0^{1,p}(\Omega,\omega_1) \longrightarrow L^{q'}(\Omega,\omega_2)$$
$$(\tilde{H}u)(x) = \mathcal{H}(x,u(x),\nabla u(x)).$$

We now show that the operator $\tilde{H}$ is bounded and continuous.

(i) Let $u \in W_0^{1,p}(\Omega,\omega_1)$. Using **(A2)** and Remaek 3.1, we obtain

$$\begin{aligned}\|\tilde{H}u\|^{q'}_{L^{q'}(\Omega,\omega_2)} &= \int_\Omega |\mathcal{H}(x,u(x),\nabla u(x))|^{p'}\omega_2 dx \\ &\le \int_\Omega \left(K_3 + h_4|u|^{\frac{q}{q'}} + h_5|\nabla u|^{\frac{q}{q'}}\right)^{q'}\omega_2 dx \\ &\le C_q \int_\Omega \left[K_3^{q'} + h_4^{q'}|u|^q + h_5^{q'}|\nabla u|^q\right]\omega_2 dx \\ &\le C_q\Big[\|K_3\|^{q'}_{L^{q'}(\Omega,\omega_2)} + \|h_4\|^{q'}_{L^\infty(\Omega)}\|u\|^q_{L^q(\Omega,\omega_2)} \\ &+ \|h_5\|^{q'}_{L^\infty(\Omega)}\|\nabla u\|^q_{L^q(\Omega,\omega_2)}\Big] \\ &\le C_q\Big[\|K_3\|^{q'}_{L^{q'}(\Omega,\omega_2)} + \|h_4\|^{q'}_{L^\infty(\Omega)}C^q_{p,q}\|u\|^q_{L^p(\Omega,\omega_1)} \\ &+ \|h_5\|^{q'}_{L^\infty(\Omega)}C^q_{p,q}\|\nabla u\|^q_{L^p(\Omega,\omega_1)}\Big] \\ &\le C_q\Big[\|K_3\|^{q'}_{L^{q'}(\Omega,\omega_2)} + C^q_{p,q}\Big(\|h_4\|^{q'}_{L^\infty(\Omega)} \\ &+ \|h_5\|^{q'}_{L^\infty(\Omega)}\Big)\|u\|^q_{W_0^{1,p}(\Omega,\omega_1)}\Big],\end{aligned}$$

where the constant C_q depends only on q.

(ii) Let $u_m \longrightarrow u$ in $W_0^{1,p}(\Omega,\omega_1)$ as $m \longrightarrow \infty$. We need to show that $\tilde{H}u_m \longrightarrow \tilde{H}u$ in $L^{q'}(\Omega,\omega_2)$.
According to **(A2)** and Remark 3.1, we have

$$\begin{aligned}&\|\tilde{H}u_{m_k} - \tilde{H}u\|^{q'}_{L^{q'}(\Omega,\omega_2)} = \int_\Omega |\tilde{H}u_{m_k}(x) - \tilde{H}u(x)|^{q'}\omega_2 dx \\ &\le \int_\Omega \Big(|\mathcal{H}(x,u_{m_k},\nabla u_{m_k})| + |\mathcal{H}(x,u,\nabla u)|\Big)^{q'}\omega_2 dx \\ &\le C_q \int_\Omega \left(|\mathcal{H}(x,u_{m_k},\nabla u_{m_k})|^{q'} + |\mathcal{H}(x,u,\nabla u)|^{q'}\right)\omega_2 dx \\ &\le C_q\Bigg[\int_\Omega \left(K_3 + h_4|u_{m_k}|^{\frac{q}{q'}} + h_5|\nabla u_{m_k}|^{\frac{q}{q'}}\right)^{q'}\omega_2 dx \\ &+ \int_\Omega \left(K_3 + h_4|u|^{\frac{q}{q'}} + h_5|\nabla u|^{\frac{q}{q'}}\right)^{q'}\omega_2 dx\Bigg] \\ &\le 2C_qC'_q \int_\Omega \left(K_3^{q'} + h_4^{q'}\Phi_1^q + h_5^{q'}\Phi_2^q\right)\omega_2 dx \\ &\le 2C_qC'_q\Big[\|K_3\|^{q'}_{L^{q'}(\Omega,\omega_2)} + \|h_4\|^{q'}_{L^\infty(\Omega)}\|\Phi_1\|^q_{L^q(\Omega,\omega_2)} \\ &+ \|h_5\|^{q'}_{L^\infty(\Omega)}\|\Phi_2\|^q_{L^q(\Omega,\omega_2)}\Big] \\ &\le 2C_qC'_q\Big[\|K_3\|^{q'}_{L^{q'}(\Omega,\omega_2)} + C^q_{p,q}\|h_4\|^{q'}_{L^\infty(\Omega)}\|\Phi_1\|^q_{L^p(\Omega,\omega_1)} \\ &+ C^q_{p,q}\|h_5\|^{q'}_{L^\infty(\Omega)}\|\Phi_2\|^q_{L^p(\Omega,\omega_1)}\Big].\end{aligned}$$

Hence, from **(A1)**, we deduce, as $k \longrightarrow \infty$

$$\tilde{H}u_{m_k}(x) \longrightarrow \tilde{H}u(x), \quad \text{a.e. } x \in \Omega.$$

Therefore, by the the Lebesgue's theorem, we obtain

$$\|\tilde{H}u_{m_k} - \tilde{H}u\|_{L^{q'}(\Omega,\omega_2)} \longrightarrow 0,$$

that is,

$$\tilde{H}u_{m_k} \longrightarrow \tilde{H}u \quad \text{in} \quad L^{q'}(\Omega,\omega_2).$$

Thanks to convergence principle in Banach spaces, we conclude that

$$\tilde{H}u_m \longrightarrow \tilde{H}u \quad \text{in} \quad L^{q'}(\Omega,\omega_2). \tag{9}$$

Finally, let $\varphi \in W_0^{1,p}(\Omega,\omega_1)$ and using Hölder inequality, Theorem 2.2(with $\theta = 1$) and Remark 3.1, we obtain

$$\begin{aligned} &|\mathbf{B}_1(u_m,\varphi) - \mathbf{B}_1(u,\varphi)| \\ &\quad = |\int_\Omega \langle \mathcal{A}(x,\nabla u_m) - \mathcal{A}(x,\nabla u), \nabla\varphi\rangle \omega_1 dx| \\ &\quad \le \sum_{j=1}^n \int_\Omega |\mathcal{A}_j(x,\nabla u_m) - \mathcal{A}_j(x,\nabla u)||D_j\varphi|\omega_1 dx \\ &\quad = \sum_{j=1}^n \int_\Omega |F_j u_m - F_j u||D_j\varphi|\omega_1 dx \\ &\quad \le \sum_{j=1}^n \|F_j u_m - F_j u\|_{L^{p'}(\Omega,\omega_1)} \|D_j\varphi\|_{L^p(\Omega,\omega_1)} \\ &\quad \le \left(\sum_{j=1}^n \|F_j u_m - F_j u\|_{L^{p'}(\Omega,\omega_1)}\right) \|\varphi\|_{W_0^{1,p}(\Omega,\omega_1)}, \end{aligned}$$

$$\begin{aligned} &|\mathbf{B}_2(u_m,\varphi) - \mathbf{B}_2(u,\varphi)| \\ &= |\int_\Omega \langle \mathcal{B}(x,u_m,\nabla u_m) - \mathcal{B}(x,u,\nabla u), \nabla\varphi\rangle \omega_2 dx| \\ &\le \sum_{j=1}^n \int_\Omega |\mathcal{B}_j(x,u_m,\nabla u_m) - \mathcal{B}_j(x,u,\nabla u)||D_j\varphi|\omega_2 dx \\ &= \sum_{j=1}^n \int_\Omega |G_j u_m - G_j u||D_j\varphi|\omega_2 dx \\ &\le \left(\sum_{j=1}^n \|G_j u_m - G_j u\|_{L^{q'}(\Omega,\omega_2)}\right) \|\nabla\varphi\|_{L^q(\Omega,\omega_2)} \\ &\le C_{p,q}\left(\sum_{j=1}^n \|G_j u_m - G_j u\|_{L^{q'}(\Omega,\omega_2)}\right) \|\nabla\varphi\|_{L^p(\Omega,\omega_1)} \\ &\le C_{p,q}\left(\sum_{j=1}^n \|G_j u_m - G_j u\|_{L^{q'}(\Omega,\omega_2)}\right) \|\varphi\|_{W_0^{1,p}(\Omega,\omega_1)}, \end{aligned}$$

$$\begin{aligned} &|\mathbf{B}_3(u_m,\varphi) - \mathbf{B}_3(u,\varphi)| \\ &\quad \le \int_\Omega |g(x,u_m) - g(x,u)||\varphi|\omega_1 dx \\ &\quad = \int_\Omega |Hu_m - Hu||\varphi|\omega_1 dx \\ &\quad \le \|Hu_m - Hu\|_{L^{p'}(\Omega,\omega_1)} \|\varphi\|_{L^p(\Omega,\omega_1)} \\ &\quad \le \|Hu_m - Hu\|_{L^{p'}(\Omega,\omega_1)} \|\varphi\|_{W_0^{1,p}(\Omega,\omega_1)}, \end{aligned}$$

and

$$\begin{aligned} &|\mathbf{B}_4(u_m,\varphi) - \mathbf{B}_4(u,\varphi)| \\ &\quad \le \int_\Omega |\mathcal{H}(x,u_m,\nabla u_m) - \mathcal{H}(x,u,\nabla u)||\varphi|\omega_2 dx \\ &\quad = \int_\Omega |\tilde{H}u_m - \tilde{H}u||\varphi|\omega_2 dx \\ &\quad \le \|\tilde{H}u_m - \tilde{H}u\|_{L^{q'}(\Omega,\omega_2)} \|\varphi\|_{L^q(\Omega,\omega_2)} \\ &\quad \le C_{p,q}\|\tilde{H}u_m - \tilde{H}u\|_{L^{q'}(\Omega,\omega_2)} \|\varphi\|_{W_0^{1,p}(\Omega,\omega_1)}. \end{aligned}$$

Hence, for all $\varphi \in W_0^{1,p}(\Omega,\omega_1)$, we have

$$\begin{aligned} &|\mathbf{B}(u_m,\varphi) - \mathbf{B}(u,\varphi)| \\ &\le |\mathbf{B}_1(u_m,\varphi) - \mathbf{B}_1(u,\varphi)| + |\mathbf{B}_2(u_m,\varphi) - \mathbf{B}_2(u,\varphi)| \\ &\quad + |\mathbf{B}_3(u_m,\varphi) - \mathbf{B}_3(u,\varphi)| + |\mathbf{B}_4(u_m,\varphi) - \mathbf{B}_4(u,\varphi)| \\ &\le \Big[\sum_{j=1}^n \Big(\|F_j u_m - F_j u\|_{L^{p'}(\Omega,\omega_1)} + C_{p,q}\|G_j u_m - G_j u\|_{L^{q'}(\Omega,\omega_2)}\Big) \\ &\quad + \|Hu_m - Hu\|_{L^{p'}(\Omega,\omega_1)} + C_{p,q}\|\tilde{H}u_m - \tilde{H}u\|_{L^{q'}(\Omega,\omega_2)}\Big] \|\varphi\|_{W_0^{1,p}(\Omega,\omega_1)}. \end{aligned}$$

Then, we get

$$\begin{aligned} &\|\mathbf{A}u_m - \mathbf{A}u\|_* \\ &\le \sum_{j=1}^n \Big(\|F_j u_m - F_j u\|_{L^{p'}(\Omega,\omega_1)} + C_{p,q}\|G_j u_m - G_j u\|_{L^{q'}(\Omega,\omega_2)}\Big) \\ &\quad + \|Hu_m - Hu\|_{L^{p'}(\Omega,\omega_1)} + C_{p,q}\|\tilde{H}u_m - \tilde{H}u\|_{L^{q'}(\Omega,\omega_2)}. \end{aligned}$$

Combining (6), (7), (8) and (9), we deduce that

$$\|\mathbf{A}u_m - \mathbf{A}u\|_* \longrightarrow 0 \text{ as } m \longrightarrow \infty,$$

that is, **A** is continuous.
Hence, the proof of the theorem 4.1 is completed.

V. Example

Let $\Omega = \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 < 1\}$, and consider the weight functions $\omega_1(x,y) = \big(x^2+y^2\big)^{-1/2}$ and $\omega_2(x,y) = \big(x^2+y^2\big)^{-1/3}$ (we have that $\omega_1, \omega_2 \in A_4$, $p = 4$ and $q = 3$), and the functions $\mathcal{B}_j : \Omega\times\mathbb{R}\times\mathbb{R}^2 \longrightarrow \mathbb{R}$, $\mathcal{A}_j : \Omega\times\mathbb{R}^2 \longrightarrow \mathbb{R}$ $(j = 1,2)$, $g : \Omega\times\mathbb{R}\times \longrightarrow \mathbb{R}$ and $\mathcal{H} : \Omega\times\mathbb{R}\times\mathbb{R}^2 \longrightarrow \mathbb{R}$ defined by

$$A_j((x,y),\xi) = h_1(x,y)\xi_j^3,$$

where $h_1(x,y) = 2e^{(x^2+y^2)}$,

$$\mathcal{B}_j((x,y),\eta,\xi) = h_3(x,y)|\xi_j|\xi_j,$$

where $h_3(x,y) = 2 + sin(x^2+y^2)$,

$$g((x,y),\eta) = h_6(x,y)|\eta|^3 sgn(\eta),$$

where $h_6(x,y) = 2 - sin^2(x+y)$, and

$$\mathcal{H}((x,y),\eta,\xi) = h_5(x,y)\xi^2 sgn(\eta),$$

where $h_5(x,y) = 2 - cos^2(xy)$.
Let us consider the partial differential operator

$$\begin{aligned} Lu(x) = -\text{div}\Big[\omega_1(x)\mathcal{A}(x,\nabla u(x)) + \omega_2(x)\mathcal{B}(x,u(x),\nabla u(x))\Big] \\ + \omega_1(x)g(x,u(x)) + \omega_2(x)\mathcal{H}(x,u(x),\nabla u(x)), \end{aligned} \tag{10}$$

Therefore, by Theorem 4.1, the problem

$$\begin{cases} Lu(x,y) = \frac{cos(xy)}{(x^2+y^2)} - \frac{\partial}{\partial x}\left(\frac{sin(xy)}{(x^2+y^2)}\right) - \frac{\partial}{\partial y}\left(\frac{sin(xy)}{(x^2+y^2)}\right) & \text{in } \Omega, \\ u(x,y) = 0 & \text{on } \partial\Omega, \end{cases}$$

admits one and only solution $u \in W_0^{1,4}(\Omega,\omega_1)$.

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