On the Cauchy problem for the fractional drift-diffusion system in critical Fourier-Besov-Morrey spaces

Achraf Azanzal *, Adil Abbassi * & Chakir Allalou *
* Laboratory of Applied Mathematics and Scientific Computing, Sultan Moulay Slimane University, B.P523, Beni Mellal, Morocco,
Email: achraf0665@gmail.com, {abbassi91,chakir.allalou}@yahoo.fr

***Abstract*—In this paper, we study the Cauchy problem of the fractional drift-diffusion system. By using the Fourier localization argument and the Littlewood Paley theory, we get the local well-posedness for large initial data in critical Fourier-Besov-Morrey space $\mathcal{FN}^{2-2\alpha+\frac{n}{p'}+\frac{\lambda}{p}}_{p,\lambda,q} \times \mathcal{FN}^{2-2\alpha+\frac{n}{p'}+\frac{\lambda}{p}}_{p,\lambda,q}$, Moreover, if the initial data is sufficiently small then the solution is global.**

***Index Terms*—Drift-diffusion, Local existence, Littlewood-Paley theory, Fourier-Besov-Morrey spaces.**

I. Introduction

In this paper, we consider the following Cauchy problem for the fractional drift-diffusion system in $\mathbb{R}^n \times \mathbb{R}^+$ with fractional Laplacian

$$\begin{cases} \partial_t v + (-\Delta)^{\frac{\alpha}{2}} v = -\nabla \cdot (v\nabla\phi) & \text{in } \mathbb{R}^n \times (0,\infty), \\ \partial_t w + (-\Delta)^{\frac{\alpha}{2}} w = \nabla \cdot (w\nabla\phi) & \text{in } \mathbb{R}^n \times (0,\infty), \\ \Delta\phi = v - w & \text{in } \mathbb{R}^n \times (0,\infty), \\ v(x,0) = v_0(x), \quad w(x,0) = w_0(x) & \text{in } \mathbb{R}^n, \end{cases} \tag{1}$$

where the unknown functions $v = v(x,t)$ and $w = w(x,t)$ denote densities of the electron and the hole in electrolytes, respectively, $\phi = \phi(x,t)$ denotes the electric potential, $v_0(x)$ and $w_0(x)$ are initial datum. Throughout this paper, we assume that $n \geq 2$ and $1 < \alpha \leq 2$.

Notice that the function ϕ is determined by the Poison equation in the third equation of (1), and it's given by:

$$\phi(x,t) = (-\Delta)^{-1}(w-v)(x,t).$$

So that the system (1) can be rewritten as the following system:

$$\begin{cases} \partial_t v + (-\Delta)^{\frac{\alpha}{2}} v = -\nabla \cdot \left(v\nabla(-\Delta)^{-1}(w-v)\right) & \text{in } \mathbb{R}^n \times \mathbb{R}^+ \\ \partial_t w + (-\Delta)^{\frac{\alpha}{2}} w = \nabla \cdot \left(w\nabla(-\Delta)^{-1}(w-v)\right) & \text{in } \mathbb{R}^n \times \mathbb{R}^+ \\ v(x,0) = v_0(x), \quad w(x,0) = w_0(x) & \text{in } \mathbb{R}^n. \end{cases} \tag{2}$$

Mathematical analysis of the Drift-diffusion system has drawn much attention during the past three decades, we refer the reader to see [1], [5] and the references therein for previous works on this system concerning existence of classical solutions and weak solutions.

In the context of Besov spaces and for $\alpha = 2$, Karch in [14] proved existence of global solution of the system (1) with small initial data in critical Besov space $\dot{\mathcal{B}}^{-2+\frac{n}{p}}_{p,\infty}(\mathbb{R}^n)$ with $\frac{n}{2} \leq p < n$. After, Deng and Li [9] showed that the system (1) is well-posed in $\dot{\mathcal{B}}^{-\frac{3}{2}}_{4,2}(\mathbb{R}^2)$, and ill-posed in $\dot{\mathcal{B}}^{-\frac{3}{2}}_{4,r}(\mathbb{R}^2)$ for $2 < r \leq \infty$. Zhao, Liu, and Cui [21] established the existence of global and local solution of the system (1) in critical Besov space $\dot{\mathcal{B}}^{-2+\frac{n}{p}}_{p,r}(\mathbb{R}^n)$ with $1 < p < 2n$ and $1 \leq r \leq \infty$.

We mention here that if w vanishes ($w = 0$) and for $\alpha = 2$, the system (1) becomes to the well-known Keller-Segel model of chemotaxis:

$$\begin{cases} \partial_t v = \Delta v - \nabla \cdot (v\nabla\phi) & \text{in } \mathbb{R}^n \times (0,\infty), \\ \Delta\phi = v & \text{in } \mathbb{R}^n \times (0,\infty), \\ v(x,0) = v_0(x), & \text{in } \mathbb{R}^n. \end{cases} \tag{3}$$

In the paper [4] the local well-posedness of the system (3) has been proved in the three-dimensional case. Iwabuchi and Nakamura [12], [13] get the global well-posednes of (3) for small initial data in the critical space

$$\dot{\mathcal{B}}^{-2+\frac{n}{p}}_{p,r}(\mathbb{R}^n)$$

with $1 \leq p < \infty$ and $1 \leq r \leq \infty$ Inspired by the work [21], The purpose of this paper is to establish the existence of local solution to (1) for large initial data and global solution for small initial data in the critical Fourrier-Besov-Morrey space

$$\mathcal{FN}^{2-2\alpha+\frac{n}{p'}+\frac{\lambda}{p}}_{p,\lambda,q} \times \mathcal{FN}^{2-2\alpha+\frac{n}{p'}+\frac{\lambda}{p}}_{p,\lambda,q}$$

Let us firstly recall the scaling property of the systems: if (v,w) solves (1) with initial data (v_0,w_0) (ϕ can be determined by (v,w)), then (v_γ, w_γ) with $(v_\gamma(x,t), w_\gamma(x,t)) := (\gamma^\alpha v(\gamma x, \gamma^\alpha t), \gamma^\alpha w(\gamma x, \gamma^\alpha t))$ is also a solution to (1) with the initial data

$$(v_{0,\gamma}(x), w_{0,\gamma}(x)) := (\gamma^\alpha v_0(\gamma x), \gamma^\alpha w_0(\gamma x)) \tag{4}$$

(ϕ_γ can be determined by (v_γ, w_γ)).

Definition 1.1: A critical space for initial data of the system (1) is any Banach space $E \subset \mathcal{S}'(\mathbb{R}^n)$ whose norm is invariant under the scaling (4) for all $\gamma > 0$, i.e

$$\|(v_{0,\gamma}(x), w_{0,\gamma}(x))\|_E \approx \|(v_0(x), w_0(x))\|_E.$$

Under these scalings, We can show that the space pair $\mathcal{FN}^{2-2\alpha+\frac{n}{p'}+\frac{\lambda}{p}}_{p,\lambda,q} \times \mathcal{FN}^{2-2\alpha+\frac{n}{p'}+\frac{\lambda}{p}}_{p,\lambda,q}$ is critical for (1) see (Remark 2.1 for details).

International Journal on Optimization and Applications
IJOA. Vol. 1, Issue No. 2, Year 2021, www.usms.ac.ma/ijoa

In order to solve the equation (1), we consider the following equivalent integral system

$$\begin{cases} v(t) = e^{-t(-\Delta)^{\frac{\alpha}{2}}} v_0 - \int_0^t e^{-(t-\tau)(-\Delta)^{\frac{\alpha}{2}}} \nabla \cdot (v \nabla \phi(\tau))\, d\tau \\ w(t) = e^{-t(-\Delta)^{\frac{\alpha}{2}}} w_0 + \int_0^t e^{-(t-\tau)(-\Delta)^{\frac{\alpha}{2}}} \nabla \cdot (w \nabla \phi(\tau))\, d\tau. \end{cases} \tag{5}$$

With $\mathcal{F}\left((-\Delta)^{\frac{\alpha}{2}} f\right)(\xi) = |\xi|^\alpha \mathcal{F} f(\xi)$.

Throughout this paper, we use $\mathcal{FN}^s_{p,\lambda,q}$ to denote the homogenous Fourier Besov-Morrey spaces, $(v,w) \in X$ to denote $(v,w) \in X \times X$ for a Banach space X $\Big($the product $X \times X$ will be endowed with the usual norm $\|(v,w)\|_{X\times X} := \|v\|_X + \|w\|_X$ $\Big)$, $\|(v,w)\|_X$ to denote $\|(v,w)\|_{X\times X}$, $V \lesssim W$ means that there exists a constant $C > 0$ such that $V \leq CW$, and p' is the conjugate of p satisfying $\frac{1}{p} + \frac{1}{p'} = 1$ for $1 \leq p \leq \infty$.

Now we present our main results as follows.

Theorem 1.1: Let $n \geq 2$, $1 < \alpha \leq 2$, $\rho_0 > \frac{\alpha}{\alpha-1}$, $\max\{n-(n+3-2\alpha)p, 0\} \leq \lambda < n$, $1 \leq p < \infty$, $q \in [1,\infty]$, $(v_0, w_0) \in \mathcal{FN}^{2-2\alpha+\frac{n}{p'}+\frac{\lambda}{p}}_{p,\lambda,q}$ *and* $\frac{1}{\rho_0} + \frac{1}{\rho_0'} = 1$.

Then there exists $T \geq 0$ such that the system (1) has a unique local solution
$(v,w) \in X_T$, where

$$X_T = \mathfrak{L}^{\rho_0}\left(0,T; \mathcal{FN}^{2-2\alpha+\frac{n}{p'}+\frac{\lambda}{p}+\frac{2}{\rho_0}}_{p,\lambda,q}\right) \cap \mathfrak{L}^{\rho_0'}\left(0,T; \mathcal{FN}^{2-2\alpha+\frac{n}{p'}+\frac{\lambda}{p}+\frac{2}{\rho_0'}}_{p,\lambda,q}\right)$$

and

$$(v,w) \in \mathcal{C}\left(0,T; \mathcal{FN}^{2-2\alpha+\frac{n}{p'}+\frac{\lambda}{p}+\frac{2}{\rho_0}}_{p,\lambda,q}\right).$$

Besides, there exixts $K \geq 0$ such that if (v_0, w_0) satisfies: $\|(v_0,w_0)\|_{\mathcal{FN}^{2-2\alpha+\frac{n}{p'}+\frac{\lambda}{p}}_{p,\lambda,q}} \leq K$, then the above assertion holds for $T = \infty$; i.e, the solution (v,w) is global.

II. Preliminaries

In this section, we give some notations and recall basic properties about Fourier-Besov-Morrey spaces that will be used throughout the paper.

The Fourier-Besov-Morrey spaces were introduced in [10] are constructed via a type of localization on Morrey spaces.

We define the function spaces M_p^λ.

Definition 2.1: [15] Let $1 \leq p \leq \infty$ and $0 \leq \lambda < n$. The homogeneous Morrey space M_p^λ is the set of all functions $f \in L^p(B(x_0,r))$ such that

$$\|f\|_{M_p^\lambda} = \sup_{x_0\in\mathbb{R}^n} \sup_{r>0} r^{-\frac{\lambda}{p}} \|f\|_{L^p(B(x_0,r))} < \infty, \tag{6}$$

where $B(x_0,r)$ is the open ball in $\mathbb{R}^n$ centered at x_0 and with radius $r > 0$.

The space M_p^λ endowed with the norm $\|f\|_{\mathrm{M}_p^\lambda}$ is a Banach space.

When $p = 1$, the L^1 -norm in (6) is understood as the total variation of the measure f on $B(x_0,r)$ and M_p^λ as a subspace of Radon measures. When $\lambda = 0$, we have $M_p^0 = L^p$.

The proofs of the results presented in this paper are based on a dyadic partition of unity in the Fourier variables, the so-called, homogeneous Littlewood-Paley decomposition. We recall briefly this construction below. For more detail, we refer the reader to [2].

Let $f \in S'(\mathbb{R}^n)$. Define the Fourier transform as

$$\hat{f}(\xi) = \mathcal{F}f(\xi) = (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} e^{-ix\cdot\xi} f(x)dx$$

and its inverse Fourier transform as

$$\check{f}(x) = \mathcal{F}^{-1}f(x) = (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} e^{ix\cdot\xi} f(\xi)d\xi.$$

Let $\varphi \in S(\mathbb{R}^d)$ be such that $0 \leq \varphi \leq 1$ and $\operatorname{supp}(\varphi) \subset \left\{\xi \in \mathbb{R}^d : \frac{3}{4} \leq |\xi| \leq \frac{8}{3}\right\}$ and

$$\sum_{j\in\mathbb{Z}} \varphi\left(2^{-j}\xi\right) = 1, \quad \text{for all } \xi \neq 0.$$

We denote

$$\varphi_j(\xi) = \varphi\left(2^{-j}\xi\right), \quad \psi_j(\xi) = \sum_{k\leq j-1} \varphi_k(\xi)$$

and

$$h(x) = \mathcal{F}^{-1}\varphi(x), \quad g(x) = \mathcal{F}^{-1}\psi(x).$$

We now present some frequency localization operators:

$$\dot{\Delta}_j f = \varphi_j(D) f = 2^{dj} \int_{\mathbb{R}^d} h\left(2^j y\right) f(x-y)dy$$

and

$$\dot{S}_j f = \sum_{k\leq j-1} \dot{\Delta}_k f = \psi_j(D) f = 2^{dj} \int_{\mathbb{R}^d} g\left(2^j y\right) f(x-y)dy.$$

From the definition, one easily derives that

$$\dot{\Delta}_j \dot{\Delta}_k f = 0, \quad \text{if } |j-k| \geq 2$$
$$\dot{\Delta}_j\left(\dot{S}_{k-1} f \dot{\Delta}_k f\right) = 0, \quad \text{if } |j-k| \geq 5.$$

The following Bony paraproduct decomposition will be applied throughout the paper.

$$uv = \dot{T}_u v + \dot{T}_v u + R(u,v)$$

where $\dot{T}_u v = \sum_{j\in\mathbb{Z}} \dot{S}_{j-1} u \dot{\Delta}_j v$, $\quad \dot{R}(u,v) = \sum_{j\in\mathbb{Z}} \dot{\Delta}_j u \tilde{\Delta}_j v$, $\tilde{\Delta}_j v = \sum_{|j'-j|\leq 1} \dot{\Delta}_{j'} v$.

Lemma 2.1: [10] Let $1 \leq p_1, p_2, p_3 < \infty$ and $0 \leq \lambda_1, \lambda_2, \lambda_3 < n$.

(i) (Hölder's inequality) Let $\frac{1}{p_3} = \frac{1}{p_1} + \frac{1}{p_2}$ and $\frac{\lambda_3}{p_3} = \frac{\lambda_1}{p_1} + \frac{\lambda_2}{p_2}$, then we have

$$\|fg\|_{\mathrm{M}_{p_3}^{\lambda_3}} \leq \|f\|_{\mathrm{M}_{p_1}^{\lambda_1}} \|g\|_{\mathrm{M}_{p_2}^{\lambda_2}}. \tag{7}$$

(ii) (Young's inequality) If $\varphi \in L^1$ and $g \in \mathrm{M}_{p_1}^{\lambda_1}$, then

$$\|\varphi * g\|_{\mathrm{M}_{p_1}^{\lambda_1}} \leq \|\varphi\|_{L^1} \|g\|_{\mathrm{M}_{p_1}^{\lambda_1}}, \tag{8}$$

where $*$ denotes the standard convolution operator.

Now, we recall the Bernstein type lemma in Fourier variables in Morrey spaces.

Lemma 2.2: [10] Let $1 \leq q \leq p < \infty$, $0 \leq \lambda_1, \lambda_2 < n$, $\frac{n-\lambda_1}{p} \leq \frac{n-\lambda_2}{q}$ and let γ be a multi-index. If $supp(\widehat{f}) \subset \{|\xi| \leq A2^j\}$, then there is a constant $C > 0$ independent of f and j such that

$$\|(i\xi)^\gamma \widehat{f}\|_{\mathrm{M}_q^{\lambda_2}} \leq C2^{j|\gamma|+j(\frac{n-\lambda_2}{q}-\frac{n-\lambda_1}{p})}\|\widehat{f}\|_{\mathrm{M}_p^{\lambda_1}}. \quad (9)$$

Then, we define the function spaces $\mathcal{FN}^s_{p,\lambda,q}(\mathbb{R}^n)$, see [10].

Definition 2.2: (Homogeneous Fourier-Besov-Morrey spaces)
Let $1 \leq p, q \leq \infty$, $0 \leq \lambda < n$ and $s \in \mathbb{R}$. The Fourier-Besov-Morrey space $\mathcal{FN}^s_{p,\lambda,q}$ is defined as the set of all distributions $f \in \mathcal{S}'\backslash\mathcal{P}$, $\mathcal{P}$ is the set of all polynomials, such that $\varphi_j \hat{f} \in \mathrm{M}^\lambda_p$, for all $j \in \mathbb{Z}$, and

$$\|f\|_{\mathcal{FN}^s_{p,\lambda,q}} \stackrel{\text{def}}{=} \begin{cases} \left(\sum_{j\in\mathbb{Z}} 2^{jsq}\left\|\varphi_j\hat{f}\right\|^q_{\mathrm{M}^\lambda_p}\right)^{\frac{1}{q}} & for \quad q<\infty \\ \sup_{j\in\mathbb{Z}} 2^{js}\left\|\varphi_j\hat{f}\right\|_{\mathrm{M}^\lambda_p} & for \quad q=\infty. \end{cases} \quad (10)$$

Note that the space $\mathcal{FN}^s_{p,\lambda,q}(\mathbb{R}^n)$ equipped with the norm (10) is a Banach space. Since $\mathrm{M}^0_p = L^p$, we have $\mathcal{FN}^s_{p,0,q} = FB^s_{p,q}$,$\mathcal{FN}^s_{1,0,q} = FB^s_{1,q} = \mathcal{B}^s_q$ and $\mathcal{FN}^{-1}_{1,0,1} = \chi^{-1}$ where $\mathcal{B}^s_q$ is the Fourier-Herz space and χ^{-1} is the Lei-Lin space [18].

Remark 2.1: The space pair $\mathcal{FN}^{2-2\alpha+\frac{n}{p'}+\frac{\lambda}{p}}_{p,\lambda,q} \times \mathcal{FN}^{2-2\alpha+\frac{n}{p'}+\frac{\lambda}{p}}_{p,\lambda,q}$ is critical for (1). For this, set $u_{0,\gamma}(\xi) = \gamma^{2-2\alpha}u_0(\gamma\xi)$, then its Fourier transform is $\widehat{u_{0,\gamma}}(\xi) = \gamma^{2-2\alpha-n}\hat{u}_0\left(\gamma^{-1}\xi\right)$.
Let

$$\begin{aligned} f_j(\xi) &\stackrel{\text{def}}{=} \varphi\left(2^{-j+[\log_2\gamma]-\log_2\gamma}\xi\right)\widehat{u_{0,\gamma}}(\xi) \\ &= \varphi\left(2^{-j+[\log_2\gamma]-\log_2\gamma}\xi\right)\gamma^{2-2\alpha-n}\hat{u}_0\left(\gamma^{-1}\xi\right) \end{aligned}$$

By change of variable, we get

$$\begin{aligned} &\|f_j\|_{\mathrm{M}^\lambda_p} \\ &= \gamma^{2-2\alpha-n}\left\|\varphi\left(2^{-j+[\log_2\gamma]-\log_2\gamma}\xi\right)\hat{u}_0\left(\gamma^{-1}\xi\right)\right\|_{\mathrm{M}^\lambda_p} \\ &= \gamma^{2-2\alpha-n}\sup_{x_0\in\mathbb{R}^n}\sup_{r>0} r^{-\frac{\lambda}{p}} \\ &\left\|\varphi\left(2^{-j+[\log_2\gamma]}\gamma\gamma^{-1}\xi\right)\widehat{u_0}\left(\gamma^{-1}\xi\right)\right\|_{L^p(B(x_0,r))} \\ &= \gamma^{2-2\alpha-n}\gamma^{\frac{n}{p}}\gamma^{\frac{-\lambda}{p}}\sup_{x_0\in\mathbb{R}^n}\sup_{r>0}\left(\gamma^{-1}r\right)^{-\frac{\lambda}{p}} \\ &\left\|\varphi\left(2^{-j+[\log_2\gamma]}\eta\right)\widehat{u_0}(\eta)\right\|_{L^p(B(\gamma^{-1}x_0,\gamma^{-1}r))} \\ &= 2^{\left(2-2\alpha+\frac{n}{p'}-\frac{\lambda}{p}\right)\log_2\gamma}\left\|\varphi\left(2^{-j+[\log_2\gamma]}\eta\right)\widehat{u_0}(\eta)\right\|_{\mathrm{M}^\lambda_p}, \end{aligned}$$

which implies

$$\begin{aligned} &\|\{2^{j(2-2\alpha+\frac{n}{p'}-\frac{\lambda}{p})}\|f_j(\xi)\|_{M^\lambda_p}\}\|_{l^q} \\ &= \|\{2^{j(2-2\alpha+\frac{n}{p'}-\frac{\lambda}{p})}2^{\log_2\gamma(2\alpha-2-\frac{n}{p'}+\frac{\lambda}{p})}\|\varphi_{j-[\log_2\gamma]}\widehat{u_0}(\xi)\|_{M^\lambda_p}\}\|_{l^q} \\ &\approx \|u_0\|_{\mathcal{FN}^{2-2\alpha+\frac{n}{p'}-\frac{\lambda}{p}}_{p,\lambda,q}} \end{aligned}$$

and since

$$\varphi_j(\xi)\widehat{u_{0,\gamma}}(\xi) = \sum_{|k-j|\leq 2}\varphi_j(\xi)f_k(\xi),$$

we can get

$$\|u_{0,\gamma}\|_{\mathcal{FN}^{2-2\alpha+\frac{n}{p'}-\frac{\lambda}{p}}_{p,\lambda,q}} \approx \|u_0\|_{\mathcal{FN}^{2-2\alpha+\frac{n}{p'}+\frac{\lambda}{p}}_{p,\lambda,q}}.$$

Similary, we have

$$\|w_{0,\gamma}\|_{\mathcal{FN}^{2-2\alpha+\frac{n}{p'}+\frac{\lambda}{p}}_{p,\lambda,q}} \approx \|w_0\|_{\mathcal{FN}^{2-2\alpha+\frac{n}{p'}+\frac{\lambda}{p}}_{p,\lambda,q}}.$$

Now, we give the definition of the mixed space-time spaces.

Definition 2.3: Let $s \in \mathbb{R}$, $1 \leq p < \infty$, $1 \leq q, \rho \leq \infty$, $0 \leq \lambda < n$, and $I = [0, T)$, $T \in (0, \infty]$. The space-time norm is defined on $u(t, x)$ by

$$\|u(t,x)\|_{\mathcal{L}^\rho(I,\mathcal{F}\dot{\mathcal{N}}^s_{p,\lambda,q})} = \Big\{\sum_{j\in\mathbb{Z}} 2^{jqs}\|\widehat{\dot{\Delta}_j u}\|^q_{L^\rho(I,\mathrm{M}^\lambda_p)}\Big\}^{1/q},$$

and denote by $\mathcal{L}^\rho(I, \mathcal{FN}^s_{p,\lambda,q})$ the set of distributions in $S'(\mathbb{R}\times\mathbb{R}^n)/\mathcal{P}$ with finite $\|.\|_{\mathcal{L}^\rho(I,\mathcal{FN}^s_{p,\lambda,q})}$ norm.
According to Minkowski inequality, it is easy to verify that

$$\begin{aligned} &L^\rho\left(I;\mathcal{FN}^s_{p,\lambda,q}\right) \hookrightarrow \mathcal{L}^\rho\left(I,\mathcal{FN}^s_{p,\lambda,q}\right), \quad \text{if } \rho\leq q, \\ &\mathcal{L}^\rho\left(I,\mathcal{FN}^s_{p,\lambda,q}\right) \hookrightarrow L^\rho\left(I;\mathcal{FN}^s_{p,\lambda,q}\right), \quad \text{if } \rho\geq q, \end{aligned}$$

where $\|u(t,x)\|_{L^\rho}\left(I;\mathcal{FN}^s_{p,\lambda,q}\right) := \left(\int_I\|u(\tau,\cdot)\|^\rho_{\mathcal{FN}^s_{p,\lambda,q}}d\tau\right)^{1/\rho}$.

At the end of this section we recall an existence and uniqueness result for an abstract operator equation in a Banach space, which will be used to prove Theorem 1.1 in the sequel. For the proof, we refer the reader to see [17] and [3].

Lemma 2.3: Let X be a Banach space with norm $\|.\|_X$ and $B : X\times X \longmapsto X$ be a bounded bilinear operator satisfying

$$\|B(u,v)\|_X \leq \eta\|u\|_X\|v\|_X$$

for all $u, v \in X$ and a constant $\eta > 0$. Then, if $0 < \varepsilon < \frac{1}{4\eta}$ and if $y \in X$ such that $\|y\|_X \leq \varepsilon$, the equation $x := y + B(x, x)$ has a solution $\overline{x}$ in X such that $\|\overline{x}\|_X \leq 2\varepsilon$. This solution is the only one in the ball $\overline{B}(0, 2\varepsilon)$. Moreover, the solution depends continuously on y in the sense: if $\|y'\|_X < \varepsilon$, $x' = y' + B(x', x')$, and $\|x'\|_X \leq 2\varepsilon$, then

$$\|\overline{x}-x'\|_X \leq \frac{1}{1-4\varepsilon\eta}\|y-y'\|_X.$$

III. Linear estimates in Fourier-Besov-Morrey spaces

In this section, we will establish some crucial estimates in the proof of Theorem 1.1. We now consider the following linear estimates for the fractional heat semigroup $\left\{e^{t\Delta}\right\}_{t\geq 0}$.

Lemma 3.1: Let I=(0, T), s ∈ ℝ, p, q, $\rho \in [1,\infty]$ and $0 \leq \lambda < n$. There exists a constant $C>0$ such that

$$\|e^{-t(-\Delta)^{\frac{\alpha}{2}}}u_0\|_{\mathcal{L}^{\rho}\left([0,T),\mathcal{FN}^{s+\frac{\alpha}{\rho}}_{p,\lambda,q}\right)} \leq C\|u_0\|_{\mathcal{FN}^{s}_{p,\lambda,q}}, \quad (11)$$

where $u_0 \in \mathcal{FN}^{s}_{p,\lambda,q}$.

proof Since supp$\varphi_j \subset \{\xi \in \mathbb{R}^n : 2^{j-1} \leq |\xi| \leq 2^{j+1}\}$, we obtain

$$\begin{aligned}\left\|\mathcal{F}\left[\Delta_j e^{-t(-\Delta)^{\frac{\alpha}{2}}}u_0\right]\right\|_{M_p^\lambda} &= \left\|\varphi_j e^{-t|\xi|^\alpha}\widehat{u_0}\right\|_{M_p^\lambda}\\ &\leq e^{-t2^{\alpha(j-1)}}\left\|\varphi_j\widehat{u}_0\right\|_{M_p^\lambda}.\end{aligned}$$

Then, by the Minkowski inequality, we have

$$\begin{aligned}&\left\|e^{-t(-\Delta)^{\frac{\alpha}{2}}}u_0\right\|_{\mathcal{L}^{\rho}\left(I;\mathcal{FN}^{s+\frac{\alpha}{\rho}}_{p,\lambda,q}\right)}\\ &\leq \left\|\left\{2^{j\left(s+\frac{\alpha}{\rho}\right)}\left(\int_0^T e^{-t\rho 2^{\alpha(j-1)}}dt\right)^{\frac{1}{\rho}}\left\|\varphi_j\widehat{u}_0\right\|_{M_p^\lambda}\right\}\right\|_{\ell^q}\\ &\leq \left\|\left\{2^{j\left(s+\frac{\alpha}{\rho}\right)}\left(\frac{1-e^{-T\rho 2^{(j-1)\alpha}}}{\rho 2^{\alpha(j-1)}}\right)^{\frac{1}{\rho}}\left\|\varphi_j\widehat{u}_0\right\|_{M_p^\lambda}\right\}\right\|_{\ell^q}\\ &\leq C\left\|u_0\right\|_{\mathcal{FN}^{s}_{p,\lambda,q}}.\end{aligned}$$

Lemma 3.2: [8] Let I=(0, T), s ∈ ℝ, p, q, $\rho \in [1,\infty]$ $0 \leq \lambda < n$ and $1 \leq r \leq \rho$.
There exists a constant $C>0$ such that

$$\left\|\int_0^t e^{(t-\tau)\Delta}f(\tau)d\tau\right\|_{\mathcal{L}^{\rho}\left(I;\mathcal{FN}^{s+\frac{2}{\rho}}_{p,\lambda,q}\right)} \leq C\|f\|_{\mathcal{L}^{r}\left(I;\mathcal{FN}^{s-2+\frac{2}{r}}_{p,\lambda,q}\right)}. \quad (12)$$

IV. Bilinear estimates in Fourier-Besov-Morrey spaces

Lemma 4.1: Let $I=(0,T)$, $s \in \mathbb{R}$, p, q $\in [1,\infty]$, $\max\{n-(n+3-2\alpha)p,0\} < \lambda < n$, $\rho_0 > \dfrac{\alpha}{\alpha-1}$ and $\frac{1}{\rho_0}+\frac{1}{\rho_0'}=1$. There exists a constant $C>0$ such that

$$\begin{aligned}\|\nabla.(f\nabla g)\|_{\mathfrak{L}^{1}\left(I;\mathcal{FN}^{2-2\alpha+\frac{n}{p'}+\frac{\lambda}{p}}_{p,\lambda,q}\right)} \leq C\Bigg[&\|f\|_{\mathfrak{L}^{\rho_0}\left(I;\mathcal{FN}^{2-2\alpha+\frac{n}{p'}+\frac{\lambda}{p}+\frac{2}{\rho_0}}_{p,\lambda,q}\right)}\\ &\times\|g\|_{\mathfrak{L}^{\rho_0'}\left(I;\mathcal{FN}^{2-\alpha+\frac{n}{p'}+\frac{\lambda}{p}+\frac{2}{\rho_0'}}_{p,\lambda,q}\right)}\\ +\|g\|_{\mathfrak{L}^{\rho_0}\left(I;\mathcal{FN}^{2-\alpha+\frac{n}{p'}+\frac{\lambda}{p}+\frac{2}{\rho_0}}_{p,\lambda,q}\right)} \times &\|f\|_{\mathfrak{L}^{\rho_0'}\left(I;\mathcal{FN}^{2-2\alpha+\frac{n}{p'}+\frac{\lambda}{p}+\frac{2}{\rho_0'}}_{p,\lambda,q}\right)}\Bigg]\end{aligned}$$

for all $f \in \mathfrak{L}^{\rho_0}\left(I;\mathcal{FN}^{2-2\alpha+\frac{n}{p'}+\frac{\lambda}{p}+\frac{2}{\rho_0}}_{p,\lambda,q}\right) \cap \mathfrak{L}^{\rho_0'}\left(I;\mathcal{FN}^{2-2\alpha+\frac{n}{p'}+\frac{\lambda}{p}+\frac{2}{\rho_0'}}_{p,\lambda,q}\right)$
and $g \in \mathfrak{L}^{\rho_0'}\left(I;\mathcal{FN}^{2-\alpha+\frac{n}{p'}+\frac{\lambda}{p}+\frac{2}{\rho_0'}}_{p,\lambda,q}\right) \cap \mathfrak{L}^{\rho_0}\left(I;\mathcal{FN}^{2-\alpha+\frac{n}{p'}+\frac{\lambda}{p}+\frac{2}{\rho_0}}_{p,\lambda,q}\right)$

Proof Applying Bony paraproduct decomposition and quasi-orthogonality property for Littlewood-Paley decomposition, for fixed j, we obtain

$$\begin{aligned}\dot{\Delta}_j(f\nabla g) &= \sum_{|k-j|\leq 4}\dot{\Delta}_j(\dot{S}_{k-1}f\dot{\Delta}_k\nabla g) + \sum_{|k-j|\leq 4}\dot{\Delta}_j(\dot{S}_{k-1}g\dot{\Delta}_k\nabla f)\\ &+ \sum_{k\geq j-3}\dot{\Delta}_j(\dot{\Delta}_k f\widetilde{\dot{\Delta}}_k\nabla g)\\ &= I_j^1+I_j^2+I_j^3\end{aligned}$$

Then, by the triangle inequalities in M_p^λ and in $l^q(\mathbb{Z})$, we have

$$\begin{aligned}\|\nabla.(f\nabla g)\|_{\mathfrak{L}^{1}\left(I;\mathcal{FN}^{2-2\alpha+\frac{n}{p'}+\frac{\lambda}{p}}_{p,\lambda,q}\right)} &\leq \|f\nabla g\|_{\mathfrak{L}^{1}\left(I;\mathcal{FN}^{3-2\alpha+\frac{n}{p'}+\frac{\lambda}{p}}_{p,\lambda,q}\right)}\\ &\leq \{\sum_{j\in\mathbb{Z}}2^{j(3-2\alpha+\frac{n}{p'}+\frac{\lambda}{p})q}\|\widehat{\dot{\Delta}_j(f\nabla g)}\|^q_{L^1(I,M_p^\lambda)}\}^{\frac{1}{q}}\\ &\leq \{\sum_{j\in\mathbb{Z}}2^{j(3-2\alpha+\frac{n}{p'}+\frac{\lambda}{p})q}\|\widehat{I_j^1}\|^q_{L^1(I,M_p^\lambda)}\}^{\frac{1}{q}}\\ &+\{\sum_{j\in\mathbb{Z}}2^{j(3-2\alpha+\frac{n}{p'}+\frac{\lambda}{p})q}\|\widehat{I_j^2}\|^q_{L^1(I,M_p^\lambda)}\}^{\frac{1}{q}}\\ &+\{\sum_{j\in\mathbb{Z}}2^{j(3-2\alpha+\frac{n}{p'}+\frac{\lambda}{p})q}\|\widehat{I_j^3}\|^q_{L^1(I,M_p^\lambda)}\}^{\frac{1}{q}}\\ &:= J_1+J_2+J_3\end{aligned}$$

By using the Young's inequality in Morrey spaces and Bernstein-type inequality with $|\gamma|=0$, we have

$$\left\|\varphi_j\hat{f}\right\|_{L^1} \leq C2^{j(\frac{n}{p'}+\frac{\lambda}{p})}\left\|\varphi_j\hat{f}\right\|_{M_p^\lambda}$$

Then

$$\|\widehat{I_j^1}\|_{L^1(I,M_p^\lambda)} \leq \sum_{|k-j|\leq 4} \|(\widehat{\dot{S}_{k-1}f\dot{\Delta}_k\nabla g})\|_{L^1(I,M_p^\lambda)}$$
$$\leq \sum_{|k-j|\leq 4} \|\varphi_j\widehat{\nabla g}\|_{L^{\rho_0'}(I,M_p^\lambda)} \sum_{l\leq k-2} \|\varphi_l\widehat{f}\|_{L^{\rho_0}(I,L^1)}$$
$$\leq C \sum_{|k-j|\leq 4} 2^k\|\varphi_k\widehat{g}\|_{L^{\rho_0'}(I,M_p^\lambda)} \sum_{l\leq k-2} 2^{(\frac{n}{p'}+\frac{\lambda}{p})l}\|\varphi_l\widehat{f}\|_{L^{\rho_0}(I,M_p^\lambda)}$$
$$\leq C \sum_{|k-j|\leq 4} 2^k\|\varphi_j\widehat{g}\|_{L^{\rho_0'}(I,M_p^\lambda)} \sum_{l\leq k-2} 2^{(2-2\alpha+\frac{n}{p'}+\frac{\lambda}{p}+\frac{\alpha}{\rho_0})l} 2^{(2\alpha-2-\frac{\alpha}{\rho_0})l}\|\varphi_l\widehat{f}\|_{L^{\rho_0}(I,M_p^\lambda)}$$
$$\leq C\|f\|_{\mathfrak{L}^{\rho_0}(I,\mathcal{FN}_{p,\lambda,q}^{2-2\alpha+\frac{n}{p'}+\frac{\lambda}{p}+\frac{\alpha}{\rho_0}})} \sum_{|k-j|\leq 4} 2^k\Big(\sum_{l\leq k-2} 2^{l(2\alpha-2-\frac{\alpha}{\rho_0})q'}\Big)^{\frac{1}{q'}}\|\varphi_k\widehat{g}\|_{L^{\rho_0'}(I,M_p^\lambda)}$$
$$\leq C\|f\|_{\mathfrak{L}^{\rho_0}(I,\mathcal{FN}_{p,\lambda,q}^{2-2\alpha+\frac{n}{p'}+\frac{\lambda}{p}+\frac{\alpha}{\rho_0}})} \sum_{|k-j|\leq 4} 2^{k(2\alpha-1-\frac{\alpha}{\rho_0})}\|\varphi_k\widehat{g}\|_{L^{\rho_0'}(I,M_p^\lambda)},$$

where we have used the fact that $\rho_0 > \dfrac{\alpha}{\alpha-1}$ in the last inequality.
Thus, by using the Young inequality, we have

$$J_1 \leq C\|f\|_{\mathfrak{L}^{\rho_0}(I,\mathcal{FN}_{p,\lambda,q}^{2-2\alpha+\frac{n}{p'}+\frac{\lambda}{p}+\frac{\alpha}{\rho_0}})} \Big(\sum_{j\in\mathbb{Z}} 2^{j(3-2\alpha+\frac{n}{p'}+\frac{\lambda}{p})q}\big(\sum_{|k-j|\leq 4} 2^{k(2\alpha-1-\frac{\alpha}{\rho_0})}\|\varphi_k\widehat{g}\|_{L^{\rho_0'}(I,M_p^\lambda)}\big)^q\Big)^{\frac{1}{q}}$$
$$\leq C\|f\|_{\mathfrak{L}^{\rho_0}(I,\mathcal{FN}_{p,\lambda,q}^{2-2\alpha+\frac{n}{p'}+\frac{\lambda}{p}+\frac{\alpha}{\rho_0}})} \Big(\sum_{j\in\mathbb{Z}}\big(\sum_{|k-j|\leq 4} 2^{(j-k)(-1+\frac{n}{p'}+\frac{\lambda}{p})q} 2^{k(2+\frac{n}{p'}+\frac{\lambda}{p}-\frac{\alpha}{\rho_0})}\|\varphi_k\widehat{g}\|_{L^{\rho_0'}(I,M_p^\lambda)}\big)^q\Big)^{\frac{1}{q}}$$
$$\leq C\|f\|_{\mathfrak{L}^{\rho_0}(I,\mathcal{FN}_{p,\lambda,q}^{2-2\alpha+\frac{n}{p'}+\frac{\lambda}{p}+\frac{\alpha}{\rho_0}})} \|g\|_{\mathfrak{L}^{\rho_0'}(I,\mathcal{FN}_{p,\lambda,q}^{2-\alpha+\frac{n}{p'}+\frac{\lambda}{p}+\frac{\alpha}{\rho_0'}})},$$

where we have used $\frac{1}{\rho_0}+\frac{1}{\rho_0'}=1$.
Similary, we get

$$J_2 \leq C\|g\|_{\mathfrak{L}^{\rho_0}(I,\mathcal{FN}_{p,\lambda,q}^{2-\alpha+\frac{n}{p'}+\frac{\lambda}{p}+\frac{\alpha}{\rho_0}})} \|f\|_{\mathfrak{L}^{\rho_0'}(I,\mathcal{FN}_{p,\lambda,q}^{2-2\alpha+\frac{n}{p'}+\frac{\lambda}{p}+\frac{\alpha}{\rho_0'}})}.$$

For J_3, first we use the Young inequality in Morrey spaces, the Bernstein inequality ($|\gamma|=0$) together with the Hölder inequality, to get

$$\|\widehat{I_j^3}\|_{L^1(I,M_p^\lambda)} \leq \sum_{k\geq j-3} \|(\widehat{\dot{\Delta}_k f\tilde{\dot{\Delta}}_k\nabla g})\|_{L^1(I,M_p^\lambda)}$$
$$= \sum_{k\geq j-3} \|(\widehat{\dot{\Delta}_k f} * \widehat{\tilde{\dot{\Delta}}_k\nabla g})\|_{L^1(I,M_p^\lambda)}$$
$$\leq \sum_{k\geq j-3} \|\varphi_k\widehat{f}\|_{L^{\rho_0'}(I,M_p^\lambda)} \sum_{|l-k|\leq 1} \|\varphi_l\widehat{\nabla g}\|_{L^{\rho_0}(I,L^1)}$$
$$\leq C \sum_{k\geq j-3} \|\varphi_k\widehat{f}\|_{L^{\rho_0'}(I,M_p^\lambda)} \sum_{|l-k|\leq 1} 2^l 2^{l(\frac{n}{p'}+\frac{\lambda}{p})}\|\varphi_l\widehat{g}\|_{L^{\rho_0}(I,M_p^\lambda)}$$
$$\leq C \sum_{k\geq j-3} \|\varphi_k\widehat{f}\|_{L^{\rho_0'}(I,M_p^\lambda)}\Big(\sum_{|l-k|\leq 1} 2^{l(\alpha-1-\frac{\alpha}{\rho_0})q'}\Big)^{\frac{1}{q'}} \|g\|_{\mathfrak{L}^{\rho_0}(I,\mathcal{FN}_{p,\lambda,q}^{2-\alpha+\frac{n}{p'}+\frac{\lambda}{p}+\frac{\alpha}{\rho_0}})}$$
$$\leq C\|g\|_{\mathfrak{L}^{\rho_0}(I,\mathcal{FN}_{p,\lambda,q}^{2-\alpha+\frac{n}{p'}+\frac{\lambda}{p}+\frac{2}{\rho_0}})} \sum_{k\geq j-3} 2^{k(\alpha-1-\frac{\alpha}{\rho_0})}\|\varphi_k\widehat{f}\|_{L^{\rho_0'}(I,M_p^\lambda)}$$

Then, applying the Hölder inequality for series, and noticing that $\lambda > n-(n+3-2\alpha)p$ implies that $3-2\alpha+\frac{n}{p'}+\frac{\lambda}{p}>0$, we obtain

$$J_3 \leq C\|g\|_{\mathfrak{L}^{\rho_0}(I,\mathcal{FN}_{p,\lambda,q}^{2-\alpha+\frac{n}{p'}+\frac{\lambda}{p}+\frac{\alpha}{\rho_0}})} \Big(\sum_{j\in\mathbb{Z}} 2^{j(3-2\alpha+\frac{n}{p'}+\frac{\lambda}{p})q}\big(\sum_{k\geq j-3} 2^{k(\alpha-1-\frac{\alpha}{\rho_0})}\|\varphi_k\widehat{f}\|_{L^{\rho_0'}(I,M_p^\lambda)}\big)^q\Big)^{\frac{1}{q}}$$
$$\leq C\|g\|_{\mathfrak{L}^{\rho_0}(I,\mathcal{FN}_{p,\lambda,q}^{2-\alpha+\frac{n}{p'}+\frac{\lambda}{p}+\frac{\alpha}{\rho_0}})} \Big(\sum_{j\in\mathbb{Z}}\big(\sum_{k\geq j-3} 2^{(j-k)(3-2\alpha+\frac{n}{p'}+\frac{\lambda}{p})} 2^{k(2-\alpha+\frac{n}{p'}+\frac{\lambda}{p}+\frac{\alpha}{\rho_0})}\|\varphi_k\widehat{f}\|_{L^{\rho_0'}(I,M_p^\lambda)}\big)^q\Big)^{\frac{1}{q}}$$
$$\leq C\|g\|_{\mathfrak{L}^{\rho_0}(I,\mathcal{FN}_{p,\lambda,q}^{2-\alpha+\frac{n}{p'}+\frac{\lambda}{p}+\frac{\alpha}{\rho_0}})} \|f\|_{\mathfrak{L}^{\rho_0'}(I,\mathcal{FN}_{p,\lambda,q}^{-2+\frac{n}{p'}+\frac{\lambda}{p}+\frac{2}{\rho_0'}})} \sum_{i\leq 3} 2^{i(3-2\alpha+\frac{n}{p'}+\frac{\lambda}{p})}$$
$$\leq C\|g\|_{\mathfrak{L}^{\rho_0}(I,\mathcal{FN}_{p,\lambda,q}^{2-\alpha+\frac{n}{p'}+\frac{\lambda}{p}+\frac{\alpha}{\rho_0}})} \|f\|_{\mathfrak{L}^{\rho_0'}(I,\mathcal{FN}_{p,\lambda,q}^{2-2\alpha+\frac{n}{p'}+\frac{\lambda}{p}+\frac{\alpha}{\rho_0'}})}.$$

Thus, we finished the proof of Lemma 4.1.

V. Proof of Theorem 1.1

To ensure the existence of the global and local solution of the system (1), we will use Lemma 2.3 with the linear and bilinear estimate that we have established in section 3 and 4.

Let $\rho_0 > \dfrac{\alpha}{\alpha-1}$ be any given real number and $\frac{1}{\rho_0}+\frac{1}{\rho_0'}=1$. Note that the space X_T defined in Theorem 1.1 is a Banach space equipped with the norm

$$\|u\|_{X_T} = \|u\|_{\mathcal{L}^{\rho_0}(I,\mathcal{FN}_{p,\lambda,q}^{2-2\alpha+\frac{n}{p'}+\frac{\lambda}{p}+\frac{\alpha}{\rho_0}})} + \|u\|_{\mathcal{L}^{\rho_0'}(I,\mathcal{FN}_{p,\lambda,q}^{2-2\alpha+\frac{n}{p'}+\frac{\lambda}{p}+\frac{\alpha}{\rho_0'}})}.$$

We first prove global existence for small initial data. For this purpose we choose $T=\infty$.
Set

$$B_1(v,w):=-\int_0^t e^{-(t-\tau)(-\Delta)^{\frac{\alpha}{2}}}\nabla\cdot\big(v\nabla(-\Delta)^{-1}(w-v)\big)(\tau)d\tau,$$

$$B_2(v,w):=\int_0^t e^{-(t-\tau)(-\Delta)^{\frac{\alpha}{2}}}\nabla\cdot\big(w\nabla(-\Delta)^{-1}(w-v)\big)(\tau)d\tau,$$

Then the equivalent integral system (1.2) can be rewritten as

$$(v(t),w(t))=(e^{-t(-\Delta)^{\frac{\alpha}{2}}}v_0,e^{-t(-\Delta)^{\frac{\alpha}{2}}}w_0)+(B_1(v,w),B_2(v,w)).\quad(13)$$

According to Lemma 3.1 with $s=2-2\alpha+\frac{n}{p'}+\frac{\lambda}{p}$, $I=[0,\infty)$ and $\rho=\rho_0$ $(or\ \ \rho_0')$, we obtain

$$\|e^{-t(-\Delta)^{\frac{\alpha}{2}}}v_0\|_{\mathfrak{L}^{\rho_0}\left(0,\infty;\mathcal{FN}_{p,\lambda,q}^{2-2\alpha+\frac{n}{p'}+\frac{\lambda}{p}+\frac{\alpha}{\rho_0}}\right)}\le C_0\,\|v_0\|_{\mathcal{FN}_{p,\lambda,q}^{2-2\alpha+\frac{n}{p'}+\frac{\lambda}{p}}}$$

and

$$\|e^{-t(-\Delta)^{\frac{\alpha}{2}}}v_0\|_{\mathfrak{L}^{\rho_0'}\left(0,\infty;\mathcal{FN}_{p,\lambda,q}^{2-2\alpha+\frac{n}{p'}+\frac{\lambda}{p}+\frac{\alpha}{\rho_0'}}\right)}\le C_0\,\|v_0\|_{\mathcal{FN}_{p,\lambda,q}^{2-2\alpha+\frac{n}{p'}+\frac{\lambda}{p}}},$$

which implies

$$\|e^{-t(-\Delta)^{\frac{\alpha}{2}}}v_0\|_{X_\infty}\le 2C_0\,\|v_0\|_{\mathcal{FN}_{p,\lambda,q}^{2-2\alpha+\frac{n}{p'}+\frac{\lambda}{p}}}$$

Similary,

$$\|e^{-t(-\Delta)^{\frac{\alpha}{2}}}w_0\|_{X_\infty}\le 2C_1\,\|w_0\|_{\mathcal{FN}_{p,\lambda,q}^{2-2\alpha+\frac{n}{p'}+\frac{\lambda}{p}}}$$

Thus

$$\|(e^{-t(-\Delta)^{\frac{\alpha}{2}}}v_0,e^{-t(-\Delta)^{\frac{\alpha}{2}}}w_0)\|_{X_\infty}\le C_2\,\|(v_0,w_0)\|_{\mathcal{FN}_{p,\lambda,q}^{2-2\alpha+\frac{n}{p'}+\frac{\lambda}{p}}}\quad(14)$$

Applying Lemma 3.2 with $s=2-2\alpha+\frac{n}{p'}+\frac{\lambda}{p}$ and $\rho_1=1$, and Lemma 4.1, we obtain

$$\begin{aligned}
&\|B_1(v,w)\|_{\mathfrak{L}^{\rho_0}\left(0,\infty;\mathcal{FN}_{p,\lambda,q}^{2-2\alpha+\frac{n}{p'}+\frac{\lambda}{p}+\frac{\alpha}{\rho_0}}\right)}\\
&=\|\int_0^t e^{-(t-\tau)(-\Delta)^{\frac{\alpha}{2}}}\nabla\cdot\big(v\nabla(-\Delta)^{-1}(w-v)\big)(\tau)d\tau\|_{\mathfrak{L}^{\rho_0}\left(0,\infty;\mathcal{FN}_{p,\lambda,q}^{2-2\alpha+\frac{n}{p'}+\frac{\lambda}{p}+\frac{\alpha}{\rho_0}}\right)}\\
&\lesssim\|\nabla\cdot\big(v\nabla(-\Delta)^{-1}(w-v)\big)\|_{\mathfrak{L}^{1}\left(0,\infty;\mathcal{FN}_{p,\lambda,q}^{2-2\alpha+\frac{n}{p'}+\frac{\lambda}{p}}\right)}\\
&\lesssim\|v\|_{\mathfrak{L}^{\rho_0}\left(0,\infty;\mathcal{FN}_{p,\lambda,q}^{2-2\alpha+\frac{n}{p'}+\frac{\lambda}{p}+\frac{\alpha}{\rho_0}}\right)}\times\|(-\Delta)^{-1}(w-v)\|_{\mathfrak{L}^{\rho_0'}\left(0,\infty;\mathcal{FN}_{p,\lambda,q}^{2-\alpha+\frac{n}{p'}+\frac{\lambda}{p}+\frac{\alpha}{\rho_0'}}\right)}\\
&+\|(-\Delta)^{-1}(w-v)\|_{\mathfrak{L}^{\rho_0}\left(0,\infty;\mathcal{FN}_{p,\lambda,q}^{2-\alpha+\frac{n}{p'}+\frac{\lambda}{p}+\frac{\alpha}{\rho_0}}\right)}\times\|v\|_{\mathfrak{L}^{\rho_0'}\left(0,\infty;\mathcal{FN}_{p,\lambda,q}^{2-2\alpha+\frac{n}{p'}+\frac{\lambda}{p}+\frac{\alpha}{\rho_0'}}\right)}\\
&\lesssim\|v\|_{\mathfrak{L}^{\rho_0}\left(0,\infty;\mathcal{FN}_{p,\lambda,q}^{2-2\alpha+\frac{n}{p'}+\frac{\lambda}{p}+\frac{\alpha}{\rho_0}}\right)}\times\|w-v\|_{\mathfrak{L}^{\rho_0'}\left(0,\infty;\mathcal{FN}_{p,\lambda,q}^{2-2\alpha+\frac{n}{p'}+\frac{\lambda}{p}+\frac{\alpha}{\rho_0'}}\right)}\\
&+\|w-v\|_{\mathfrak{L}^{\rho_0}\left(0,\infty;\mathcal{FN}_{p,\lambda,q}^{2-2\alpha+\frac{n}{p'}+\frac{\lambda}{p}+\frac{\alpha}{\rho_0}}\right)}\times\|v\|_{\mathfrak{L}^{\rho_0'}\left(0,\infty;\mathcal{FN}_{p,\lambda,q}^{2-2\alpha+\frac{n}{p'}+\frac{\lambda}{p}+\frac{\alpha}{\rho_0'}}\right)}\\
&\le C_3\Big(\|(v,w)\|_{\mathfrak{L}^{\rho_0}\left(0,\infty;\mathcal{FN}_{p,\lambda,q}^{2-2\alpha+\frac{n}{p'}+\frac{\lambda}{p}+\frac{\alpha}{\rho_0}}\right)}\times\|(v,w)\|_{\mathfrak{L}^{\rho_0'}\left(0,\infty;\mathcal{FN}_{p,\lambda,q}^{2-2\alpha+\frac{n}{p'}+\frac{\lambda}{p}+\frac{\alpha}{\rho_0'}}\right)}\Big)\\
&\le C_3\|(v,w)\|_{X_\infty}^2
\end{aligned}$$

Analogously, we get

$$\|B_1(v,w)\|_{\mathfrak{L}^{\rho_0'}\left(0,\infty;\mathcal{FN}_{p,\lambda,q}^{2-2\alpha+\frac{n}{p'}+\frac{\lambda}{p}+\frac{\alpha}{\rho_0'}}\right)}\le C_3\|(v,w)\|_{X_\infty}^2$$

Thus, we obtain

$$\|B_1(v,w)\|_{X_\infty}\le 2C_3\|(v,w)\|_{X_\infty}^2$$

Similary,

$$\|B_2(v,w)\|_{X_\infty}\le 2C_4\|(v,w)\|_{X_\infty}^2$$

Finally,

$$\|(B_1(v,w),B_2(v,w))\|_{X_\infty}\le C\|(v,w)\|_{X_\infty}^2$$

By Lemma 2.3, we know that if $\|(e^{-t(-\Delta)^{\frac{\alpha}{2}}}v_0, e^{-t(-\Delta)^{\frac{\alpha}{2}}}w_0)\|_{X_\infty} \leq \varepsilon$ with $\varepsilon = \frac{1}{4C}$, then the system (1) has a unique global solution in $\bar{B}(0,2\varepsilon) = \{x \in X_\infty \ \|x\|_{X_\infty} \leq 2\varepsilon\}$. To prove $\|(e^{-t(-\Delta)^{\frac{\alpha}{2}}}v_0, e^{-t(-\Delta)^{\frac{\alpha}{2}}}w_0)\|_{X_\infty} \leq \varepsilon$, according to (14) we have

$$\|(e^{-t(-\Delta)^{\frac{\alpha}{2}}}v_0, e^{-t(-\Delta)^{\frac{\alpha}{2}}}w_0)\|_{X_\infty} \leq C_2\left\|(v_0,w_0)\right\|_{\mathcal{FN}_{p,\lambda,q}^{2-2\alpha+\frac{n}{p'}+\frac{\lambda}{p}}}$$

So, if $\|(v_0,w_0)\|_{\mathcal{FN}_{p,\lambda,q}^{2-2\alpha+\frac{n}{p'}+\frac{\lambda}{p}}} \leq K$ with $K = \frac{1}{4CC_2}$, then (1) has a unique global solution $(v,w) \in X_\infty$ satisfying

$$\|(v,w)\|_{X_\infty} \leq \frac{1}{2C}$$

For the local existence, we shall decompose the initial data v_0 into two terms

$$v_0 = \mathcal{F}^{-1}\left(\chi_{B(0,\delta)}\hat{v}_0\right) + \mathcal{F}^{-1}\left(\chi_{B^C(0,\delta)}\hat{v}_0\right) := v_{0,1} + v_{0,2},$$

where $\delta = \delta(v_0) > 0$ is a real number. Similary, we decompose w_0

$$w_0 = \mathcal{F}^{-1}\left(\chi_{B(0,\delta)}\hat{w}_0\right) + \mathcal{F}^{-1}\left(\chi_{B^C(0,\delta)}\hat{w}_0\right) := w_{0,1} + w_{0,2}.$$

Since

$$\begin{cases} v_{0,2} \longrightarrow 0 \ \text{in} \ \mathcal{FN}_{p,\lambda,q}^{2-2\alpha+\frac{n}{p'}+\frac{\lambda}{p}} \ \text{when} \quad \delta \to +\infty, \\ w_{0,2} \longrightarrow 0 \ \text{in} \ \mathcal{FN}_{p,\lambda,q}^{2-2\alpha+\frac{n}{p'}+\frac{\lambda}{p}} \ \text{when} \quad \delta \to +\infty, \end{cases}$$

then, there exists δ large enough such that

$$C_2\|(v_{0,2},w_{0,2})\|_{\mathcal{FN}_{p,\lambda,q}^{2-2\alpha+\frac{n}{p'}+\frac{\lambda}{p}}} \leq \frac{\varepsilon}{2}.$$

We get

$$\left\|\left(e^{-t(-\Delta)^{\frac{\alpha}{2}}}v_0, e^{-t(-\Delta)^{\frac{\alpha}{2}}}w_0\right)\right\|_{X_T} \leq \frac{\varepsilon}{2} + \left\|\left(e^{-t(-\Delta)^{\frac{\alpha}{2}}}v_{0,1}, e^{-t(-\Delta)^{\frac{\alpha}{2}}}w_{0,1}\right)\right\|_{X_T}$$

We have

$$\begin{aligned}\left\|\left(e^{-t(-\Delta)^{\frac{\alpha}{2}}}v_{0,1}, e^{-t(-\Delta)^{\frac{\alpha}{2}}}w_{0,1}\right)\right\|_{X_T} &= \\ \left\|\left(e^{-t(-\Delta)^{\frac{\alpha}{2}}}v_{0,1}, e^{-t(-\Delta)^{\frac{\alpha}{2}}}w_{0,1}\right)\right\|_{\mathcal{L}^{\rho_0}(I,\mathcal{FN}_{p,\lambda,q}^{2-2\alpha+\frac{n}{p'}+\frac{\lambda}{p}+\frac{\alpha}{\rho_0}})} \\ +\left\|\left(e^{-t(-\Delta)^{\frac{\alpha}{2}}}v_{0,1}, e^{-t(-\Delta)^{\frac{\alpha}{2}}}w_{0,1}\right)\right\|_{\mathcal{L}^{\rho_0'}(I,\mathcal{FN}_{p,\lambda,q}^{2-2\alpha+\frac{n}{p'}+\frac{\lambda}{p}+\frac{\alpha}{\rho_0'}})}\end{aligned}$$

Using the fact that $|\xi| \approx 2^j$ for all $j \in \mathbb{Z}$, we have

$$\begin{aligned}&\left\|\left(e^{-t(-\Delta)^{\frac{\alpha}{2}}}v_{0,1}, e^{-t(-\Delta)^{\frac{\alpha}{2}}}w_{0,1}\right)\right\|_{\mathcal{L}^{\rho_0}(I,\mathcal{FN}_{p,\lambda,q}^{2-2\alpha+\frac{n}{p'}+\frac{\lambda}{p}+\frac{\alpha}{\rho_0}})} \\ &= \Big\{\sum_{j\in\mathbb{Z}} 2^{j(2-2\alpha+\frac{n}{p'}+\frac{\lambda}{p}+\frac{\alpha}{\rho_0})q}\|\varphi_j \widehat{e^{-t(-\Delta)^{\frac{\alpha}{2}}}v_{0,1}}\|^q_{L^{\rho_0}(I,\mathrm{M}_p^\lambda)}\Big\}^{1/q} \\ &+\Big\{\sum_{j\in\mathbb{Z}} 2^{j(2-2\alpha+\frac{n}{p'}+\frac{\lambda}{p}+\frac{\alpha}{\rho_0})q}\|\varphi_j \widehat{e^{-t(-\Delta)^{\frac{\alpha}{2}}}w_{0,1}}\|^q_{L^{\rho_0}(I,\mathrm{M}_p^\lambda)}\Big\}^{1/q} \\ &= \Big\{\sum_{j\in\mathbb{Z}} 2^{j(2-2\alpha+\frac{n}{p'}+\frac{\lambda}{p})q}2^{j(\frac{\alpha}{\rho_0})q}\|\varphi_j|\xi|^\alpha\chi_{B(0,\delta)}\hat{v}_0\|^q_{L^{\rho_0}(I,\mathrm{M}_p^\lambda)}\Big\}^{1/q} \\ &+\Big\{\sum_{j\in\mathbb{Z}} 2^{j(2-2\alpha+\frac{n}{p'}+\frac{\lambda}{p})q}2^{j(\frac{\alpha}{\rho_0})q}\|\varphi_j|\xi|^\alpha\chi_{B(0,\delta)}\hat{w}_0\|^q_{L^{\rho_0}(I,\mathrm{M}_p^\lambda)}\Big\}^{1/q} \\ &\lesssim \delta^{\alpha+\frac{\alpha}{\rho_0}}\Big(\big(\sum_{j\in\mathbb{Z}} 2^{j(2-2\alpha+\frac{n}{p'}+\frac{\lambda}{p})q}\|\varphi_j\hat{v}_0\|^q_{L^{\rho_0}(I,\mathrm{M}_p^\lambda)}\Big)^{1/q} \\ &+\Big\{\sum_{j\in\mathbb{Z}} 2^{j(2-2\alpha+\frac{n}{p'}+\frac{\lambda}{p})q}\|\varphi_j\hat{w}_0\|^q_{L^{\rho_0}(I,\mathrm{M}_p^\lambda)}\Big\}^{1/q}\Big) \\ &\leq C_5\delta^{\alpha+\frac{\alpha}{\rho_0}}T^{\frac{1}{\rho_0}}\left\|(v_0,w_0)\right\|_{\mathcal{FN}_{p,\lambda,q}^{2-2\alpha+\frac{n}{p'}+\frac{\lambda}{p}}}\end{aligned}$$

Thus

$$\left\|\left(e^{-t(-\Delta)^{\frac{\alpha}{2}}}v_{0,1}, e^{-t(-\Delta)^{\frac{\alpha}{2}}}w_{0,1}\right)\right\|_{\mathcal{L}^{\rho_0}(I,\mathcal{FN}_{p,\lambda,q}^{2-2\alpha+\frac{n}{p'}+\frac{\lambda}{p}+\frac{\alpha}{\rho_0}})} \leq C_5\delta^{\alpha+\frac{\alpha}{\rho_0}}T^{\frac{1}{\rho_0}}\left\|(v_0,w_0)\right\|_{\mathcal{FN}_{p,\lambda,q}^{2-2\alpha+\frac{n}{p'}+\frac{\lambda}{p}}}$$

Similary,

$$\left\|\left(e^{-t(-\Delta)^{\frac{\alpha}{2}}}v_{0,1}, e^{-t(-\Delta)^{\frac{\alpha}{2}}}w_{0,1}\right)\right\|_{\mathcal{L}^{\rho_0'}(I,\mathcal{FN}_{p,\lambda,q}^{2-2\alpha+\frac{n}{p'}+\frac{\lambda}{p}+\frac{\alpha}{\rho_0'}})} \leq C_5\delta^{\alpha+\frac{\alpha}{\rho_0'}}T^{\frac{1}{\rho_0'}}\left\|(v_0,w_0)\right\|_{\mathcal{FN}_{p,\lambda,q}^{2-2\alpha+\frac{n}{p'}+\frac{\lambda}{p}}}.$$

Hence,

$$\begin{aligned}\left\|\left(e^{-t(-\Delta)^{\frac{\alpha}{2}}}v_{0,1}, e^{-t(-\Delta)^{\frac{\alpha}{2}}}w_{0,1}\right)\right\|_{X_T} &\leq \\ C_5\delta^{\alpha+\frac{\alpha}{\rho_0}}T^{\frac{1}{\rho_0}}\left\|(v_0,w_0)\right\|_{\mathcal{FN}_{p,\lambda,q}^{2-2\alpha+\frac{n}{p'}+\frac{\lambda}{p}}} \\ +C_5\delta^{\alpha+\frac{\alpha}{\rho_0'}}T^{\frac{1}{\rho_0'}}\left\|(v_0,w_0)\right\|_{\mathcal{FN}_{p,\lambda,q}^{2-2\alpha+\frac{n}{p'}+\frac{\lambda}{p}}}\end{aligned}$$

Then, if we choose T small enough such that

$$\begin{cases} C_5\delta^{\alpha+\frac{\alpha}{\rho_0}}T^{\frac{1}{\rho_0}}\left\|(v_0,w_0)\right\|_{\mathcal{FN}_{p,\lambda,q}^{2-2\alpha+\frac{n}{p'}+\frac{\lambda}{p}}} \leq \frac{\varepsilon}{4} \\ and \\ C_5\delta^{\alpha+\frac{\alpha}{\rho_0'}}T^{\frac{1}{\rho_0'}}\left\|(v_0,w_0)\right\|_{\mathcal{FN}_{p,\lambda,q}^{2-2\alpha+\frac{n}{p'}+\frac{\lambda}{p}}} \leq \frac{\varepsilon}{4}. \end{cases}$$

i.e,

$$\begin{cases} T \leq \left(\dfrac{\varepsilon}{4C_5\delta^{\alpha+\frac{\alpha}{\rho_0}} \|(v_0,w_0)\|_{\mathcal{FN}^{2-2\alpha+\frac{n}{p'}+\frac{\lambda}{p}}_{p,\lambda,q}}} \right)^{\rho_0} \\ and \\ T \leq \left(\dfrac{\varepsilon}{4C_5\delta^{\alpha+\frac{\alpha}{\rho_0'}} \|(v_0,w_0)\|_{\mathcal{FN}^{2-2\alpha+\frac{n}{p'}+\frac{\lambda}{p}}_{p,\lambda,q}}} \right)^{\rho_0'} \end{cases}$$

So, if we choose

$$T \leq \min\Bigg(\Big(\frac{\varepsilon}{4C_5\delta^{\alpha+\frac{\alpha}{\rho_0}} \|(v_0,w_0)\|_{\mathcal{FN}^{2-2\alpha+\frac{n}{p'}+\frac{\lambda}{p}}_{p,\lambda,q}}} \Big)^{\rho_0}, \Big(\frac{\varepsilon}{4C_5\delta^{\alpha+\frac{\alpha}{\rho_0'}} \|(v_0,w_0)\|_{\mathcal{FN}^{2-2\alpha+\frac{n}{p'}+\frac{\lambda}{p}}_{p,\lambda,q}}} \Big)^{\rho_0'} \Bigg)$$

then

$$\left\| \left(e^{-t(-\Delta)^{\frac{\alpha}{2}}} v_{0,1}, e^{-t(-\Delta)^{\frac{\alpha}{2}}} w_{0,1} \right) \right\|_{X_T} \leq \frac{\varepsilon}{2}.$$

This result with (5.2) yields that

$$\left\| \left(e^{-t(-\Delta)^{\frac{\alpha}{2}}} v_0, e^{-t(-\Delta)^{\frac{\alpha}{2}}} w_0 \right) \right\|_{X_T} \leq \varepsilon.$$

Thus for any arbitrary $(v_0, w_0) \in \mathcal{FN}^{2-2\alpha+\frac{n}{p'}+\frac{\lambda}{p}}_{p,\lambda,q}$, (1) has a unique local solution in $\bar{B}(0, 2\varepsilon) = \{x \in X_T \ : \ \|x\|_{X_T} \leq 2\varepsilon\}$.

Regularity:

We know if $(v, w) \in X_T \times X_T$ is a solution of (1), then we can show that

$$\nabla \cdot (v\nabla\phi), \ \ \nabla \cdot (w\nabla\phi) \in \mathfrak{L}^1 \left(0, T; \mathcal{FN}^{2-2\alpha+\frac{n}{p'}+\frac{\lambda}{p}}_{p,\lambda,q} \right).$$

By using the definition of the Fourier-Besov-Morrey spaces, we have

$$\begin{aligned} &\|v(t_1) - v(t_2)\|^q_{\mathcal{FN}^{2-2\alpha+\frac{n}{p'}+\frac{\lambda}{p}}_{p,\lambda,q}} \\ \leq &\sum_{j\leq N} \left(2^{j(2-2\alpha+\frac{n}{p'}+\frac{\lambda}{p})} \|\hat{v}_j(t_1) - \hat{v}_j(t_2)\|_{\mathrm{M}^\lambda_p} \right)^q \\ &+2 \sum_{j>N} \left(2^{j(2-2\alpha+\frac{n}{p'}+\frac{\lambda}{p})} \|\hat{v}_j(t)\|_{L^\infty(I,\mathrm{M}^\lambda_p)} \right)^q, \end{aligned}$$

where $\hat{v}_j = \varphi_j \hat{v}$. For any small constant $\varepsilon > 0$, let N be large enough such that

$$\sum_{j>N} 2^{j(2-2\alpha+\frac{n}{p'}+\frac{\lambda}{p})q} \|\hat{v}_j(t)\|^q_{L^\infty(I,\mathrm{M}^\lambda_p)} \leq \frac{\varepsilon}{4}.$$

According to Taylor's formula and using the same arguments as [[21], Proposition 2.3], we get

$$\begin{aligned} &\sum_{j\leq N} \left(2^{j(2-2\alpha+\frac{n}{p'}+\frac{\lambda}{p})} \|\hat{v}_j(t_1) - \hat{v}_j(t_2)\|_{\mathrm{M}^\lambda_p} \right)^q \\ &\lesssim |t_1 - t_2|^q \sum_{j\leq N} 2^{j(2-2\alpha+\frac{n}{p'}+\frac{\lambda}{p})q} \left\| (\hat{\partial_t u})_j \right\|^q_{L^1(I,\mathrm{M}^\lambda_p)} \\ &\lesssim |t_1 - t_2|^q \times \|\partial_t u\|^q_{\mathfrak{L}^1\left(0,T;\mathcal{FN}^{2-2\alpha+\frac{n}{p'}+\frac{\lambda}{p}}_{p,\lambda,q}\right)} \\ &\lesssim |t_1 - t_2|^q \times \Big(\|\Delta v\|^q_{\mathfrak{L}^1\left(0,T;\mathcal{FN}^{2-2\alpha+\frac{n}{p'}+\frac{\lambda}{p}}_{p,\lambda,q}\right)} + \|\nabla \cdot (v\nabla\phi)\|^q_{\mathfrak{L}^1\left(0,T;\mathcal{FN}^{2-2\alpha+\frac{n}{p'}+\frac{\lambda}{p}}_{p,\lambda,q}\right)} \Big) \\ &\lesssim |t_1 - t_2|^q \times \Big(\|v\|^q_{\mathfrak{L}^1\left(0,T;\mathcal{FN}^{2-\alpha+\frac{n}{p'}+\frac{\lambda}{p}}_{p,\lambda,q}\right)} + \|\nabla \cdot (v\nabla\phi)\|^q_{\mathfrak{L}^1\left(0,T;\mathcal{FN}^{2-2\alpha+\frac{n}{p'}+\frac{\lambda}{p}}_{p,\lambda,q}\right)} \Big) \\ &\lesssim |t_1 - t_2|^q \times \Big(\|v_0\|^q_{\mathcal{FN}^{2-2\alpha+\frac{n}{p'}+\frac{\lambda}{p}}_{p,\lambda,q}} + 2\|\nabla \cdot (v\nabla\phi)\|^q_{\mathfrak{L}^1\left(0,T;\mathcal{FN}^{2-2\alpha+\frac{n}{p'}+\frac{\lambda}{p}}_{p,\lambda,q}\right)} \Big). \end{aligned}$$

Thus, we obtain the continuity of v in time t.
Similary, we use the same discusion to get the continuity of w in time t.
Hence $(v, w) \in C\left(0, T; \mathcal{FN}^{2-2\alpha+\frac{n}{p'}+\frac{\lambda}{p}}_{p,\lambda,q}\right)$, and we are done.

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