Existance results for a neutral functionnal integrodifferential inclusion with finite delay

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***Abstract*— This paper is mainly concerned with existence of mild solution for a neutral functionnal integrodiferential inclusion with finit delay. The results are bobtained by using a fixed point theoreme for condensing multivalued maps.**

***Index Terms*—integrodifferentional inclusion, Selection, Fixed point theory, neutral functional differential and integrodifferential inclusion, convex multivalued map.**

I. INTRODUCTION:

In this paper we prove the existence of mild solution, for a neutral functional integrodifferentional inclusion with finite delay. In section 2 we will recall briefly some basic definitions and preliminary facts which will be used in the following section. Section 3 deals with the existence of mild solution for a neutral functional integrodifferentional inclusion with finite delay of the forme :

$$\begin{cases} \dfrac{d}{dt}\mathcal{F}(t,u_t) \in A\mathcal{F}(t,u_t) + \displaystyle\int_0^t B(t-s)\mathcal{F}(s,u_s)ds \\ +G(t,u_t) \quad \text{for} \quad t \in [0,b] \\ u_0 = \varphi(\theta) \quad \text{for} \quad \theta \in J_0 = [-r,0], \end{cases} \tag{1}$$

where $\big(A, \mathcal{D}(A)\big)$ is the infinitesimal generator of a compact resolvant operator $R(t)$, $t \geq 0$, in Banach space X, for $t \geq 0$ $B(t)$ is a closed linear operator with domain $\mathcal{D}(B)$, such that $\mathcal{D}(A) \subset \mathcal{D}(B)$. $G : J \times \mathcal{C}(J_0, X) \longrightarrow 2^X$ ($J_0 = [-r, 0]$), is a bounded, closed, convex, multivalued map and X a real Banach space.

For any continus function u defined on $J_1 = [-r, b]$, and any $t \in J$, we denot by u_t the element of $\mathcal{C}(J_0, X)$ defined by:

$$u_t(\theta) = u(t+\theta) = \varphi(\theta), \quad \theta \in J_0 = [-r, 0],$$

Here $u_t(.)$ represents the history of the state from time $t-r$, up to the present time t, and $\mathcal{F} : J \times \mathcal{C}(J_0, X) \longrightarrow X$ defined by :

$$\mathcal{F}(t,\varphi) = \varphi(0) - F(t,\varphi) = u(t) - F(t,u_t), \quad \forall (t,\varphi) \in J \times \mathcal{C}(J_0, X),$$

Where $F : J \times \mathcal{C}(J_0, X) \longrightarrow X$.
Wen $B = 0$ we refer to the paper of K.HILAL and K.EZZINBI [1] and the paper of K.EZZINBI and X.FU [2].
This paper is motivated by the recents results of [1] and BENCHOHRA [3]. Here we compose the above results and prove the existence of mild solution for our probleme (1), relying on a fixed point theorem for condensing maps due to Martelli [4].

II. PRELIMINARIES:

In this section, we introduce some basic definitions, notations, and lemmas that are used throughout this paper.

$\mathcal{C}(J, X)$ is the Banach space of continuous functions from J into X with the norm :

$$\|u\|_\infty := \sup\{|u(t)|; t \in J\}$$

A measurable function $u : J \longrightarrow X$ is Bochner integrable if and only if $|u|$ is Lebesgue integrable (For properties of the Bochner integral see Yosida [5]).
$L^1(J, X)$ denotes the Banach space of continuous functions $u : J \longrightarrow X$ which are Bochner integrable normed by :

$$\|u\|_{L^1} := \int_0^T |u(t)|dt \quad \text{for all } u \in L^1(J, X)$$

Lemma 2.1: :
Let $(X, \|.\|)$ be a Banach space. A multivalued map $G : X \longrightarrow 2^X$ is convex closed, if $G(x)$ is convex closed, for all $x \in X$; and G is bounded on bounded sets, if $G(B) = \underset{x\in B}{\cup} G(x)$ is bounded in X, for any bounded set B of X.

Theorem 2.1: :
G is said to be completely continuous if $G(B)$ is relatively compact, for every bounded subset $B \subset X$.

Theorem 2.2: :
G is called upper semi-continuous (u.s.c) on X, if for each $x \in X$, the set $G(x)$ is a nonempty, closed subset of X, and if for each open set B of X containing $G(x)$, there exists an open neighborhood V of x such that $G(V) \in B$.

Lemma 2.2: :
If the multivalued map G is completely continuous with nonempty compact values, then G is u.s.c. if and only if G has a closed graph $\Big($i.e $x_n \longrightarrow x, y_n \longrightarrow y; y_n \in G(x_n)$ imply $y \in G(x)\Big)$.

Definition 2.1: :
an upper semi-continuous multivalued map $G : X \longrightarrow X$ is said to be condensing if for any subset $B \subset X$ with $\alpha(B) \neq 0$, we have $\alpha\big(G(B)\big) < \alpha\big(B\big)$, where α denotes the Kuratowski measure of noncompactness [6].

Lemma 2.3: :

A completely continuous multivalued map is a condensing map

Theorem 2.3: :(Arzela–Ascoli's theorem)

Let K be a compact space and (E, d) a metric space. $A \subset \mathcal{C}(K, E)$ is relatively compact (i.e. included in a compact) if and only if, for any x of K:

- A is equicontinuous in x, i.e. for everything $\varepsilon > 0$, there exist a neighborhood V of x such that : $\forall f \in A, \forall y \in V \quad d(f(x), f(y)) < \varepsilon$
- The set $A(x) = \{f(x); f \in A\}$ is relatively compact.

In the following $\mathcal{BCC}(X)$ denotes the set of all nonempty bounded, closed and convex subsets of X

Theorem 2.4: :(Leray-schauder's fixed point)

Let X be a Banach space and $N : X \longrightarrow \mathcal{BCC}(X)$ an u.s.c condensing map. If the set :

$$\Omega := \{u \in X : \lambda u \in Nu \quad \text{for} \quad \lambda > 1\}$$

is bounded, then N has a fixed point.

Definition 2.2: :(Finite delay differential equation)

Let $r > 0$; and $\mathcal{C}_r = \mathcal{C}\big([-r, 0], \mathbb{R}^n\big)$, the Banach space of continuous functions,
$\varphi : [-r, 0] \longrightarrow \mathbb{R}^n$ with $\|\varphi\|_\infty = \sup_{\theta \in [-r,0]} \|\varphi(\theta)\|$. We denot by u_t element of $\mathcal{C}_r$ defined by :

$$u_t(\theta) = u(t + \theta) = \varphi(\theta), \quad \theta \in J_0 = [-r, 0],$$

Let $f : \mathbb{R}^+ \times \mathcal{C}_r \longrightarrow \mathbb{R}$, a general form of the finit-delay differential equation is :

$$\frac{d}{dt} u(t) = f(t, u_t)$$

Definition 2.3: :(Resolvent operator [2])

A family of bounded linear operators $R(t) \in B(X)$, $\big(B(X)$ is the Banach space of all linear bounded operator from X into $X\big)$, for $t \in J$ is called a resolvent operator for :

$$\frac{du}{dt} = Au(t) + \int_0^t f(t - s)u(t)ds$$

If:

1- $R(0) = I$, the identity operator on X, and $\|R(t)\| \leqslant M$ with $M > 1$.
2- For all $u \in X$; $R(t)u$ is continuous for $t \in J$
3- $R(t) \in B(Y)$; $t \in J$; where Y is the Banach space formed from $\mathcal{D}(A)$, for $y \in Y, R(.)y \in \mathcal{C}^1(J, X) \cap \mathcal{C}(J, Y)$ and :

$$R^{'}(t)y = AR(t)y + \int_0^t f(t - s)R(s)yds = R(t)Ay + \int_0^t R(t - s)f(s)yds.$$

III. EXISTANCE RESULTS :

In order to define the concept of mild solution for (1), by comparaison with the evolution problem

$$\frac{dv}{dt} = Av(t) + \int_0^t f(t - s)v(t)ds + h(t) \quad ; \quad v(0) = a$$

We asssociate (1) to the integral equation :

$$u(t) = R(t)\mathcal{F}(0,\ \varphi) + F(t,\ u_t) + \int_0^t R(t - s)g(s)\mathrm{d}s \quad t \in [0, b]$$

Where $g \in \mathcal{S}_{G,u} = \{g \in L^1(J, X) : g(t) \in G(t, u_t); \quad t \in J\}$

Definition 3.1: :

A function $u \in \mathcal{C}([-r, b], X)$ is called a mild solution of (1) if :

1- $u(0) = \varphi(\theta); \quad \theta \in [-r, 0]$.
2- There exist a function $g \in \mathcal{S}_{G,u}$ such that :

$$u(t) = R(t)\mathcal{F}(0,\ \varphi) + F(t,\ u_t) + \int_0^t R(t - s)g(s)\mathrm{d}s \quad t \in [0, b]$$

Where, $\mathcal{F}(0,\ \varphi) = \varphi(\theta) - F(t, \varphi)$

Assume that :

(H1)- A is the infinitesimal generator of a compact resolvent operator $R(t)$ in X such that :
$$\|R(t)\| \leq M_1 \quad for \quad some \quad M_1 \geq 1 \quad ; \quad t \in J$$
(H2)- There exists constants $0 \leq c_1 < 1$ and $c_2 \geq 0$ such that :
$$|F(t, u)| \leq c_1\|u\| + c_2; \quad t \in J \quad u \in \mathcal{C}(J_0, X)$$
(H3)- $\varphi \in \mathcal{C}([-r, 0], X)$ is completely continuous and there exists a constant M_2 such that:
$$\|\varphi\| \leq M_2$$
(H4)- $G : J \times \mathcal{C}(J_0, X) \longrightarrow \mathcal{BCC}(X)$; $(t, u) \longrightarrow G(t, u)$ is measurable with respect to t for each $u \in \mathcal{C}(J_0, X)$, u.s.c with respect to u for each $t \in J$; and for each fixed $u \in \mathcal{C}(J_0, X)$ the set :
$$\mathcal{S}_{G,u} = \{g \in L^1(J, X) : g(t) \in G(t, u_t); \quad t \in J\}$$
is nonempty.
(H5)- $\|G(t, u)\| := sup\{|g| : g \in G(t, u)\} \leq p(t)\Psi(\|u\|)$ for all $t \in J$ and all $u \in \mathcal{C}(J_0, X)$, where $p \in L^1(J, \mathbb{R}^+)$ and $\Psi : \mathbb{R}^+ \longrightarrow [0, +\infty)$ is continuous and increasing with :
$$\int_0^b \omega(s)ds < \int_c^\infty \frac{d\tau}{\tau + \Psi(\tau)}$$
Where
$$c = \frac{1}{1 - c_1}\{M_1\big(M_2(1 + c_1) + c_2\big) + c_2\} \quad and \quad \omega(s) = \frac{1}{1 - c_1} M_1 p(t)$$
(H6)- The function F is complety continuous and for any bounded set $B \subseteq \mathcal{C}(J_1, X)$ the set $\{t \longrightarrow F(t, u_t) : u \in B\}$ is equicontinuous in $\mathcal{C}$.

The following lemma is crucial in the proof of our existence results.

Lemma 3.1: :

Let I be a compact real interval and X be a Banach space. Let G be a multivalued map satisfying $(H4)$. And let Γ be a linear continuous mapping from $L^1(I, X)$ to $\mathcal{C}(J, X)$. Then the operator :

$$\Gamma \circ \mathcal{S}_G : \mathcal{C}(I, X) \longrightarrow \mathcal{BCC}\big(\mathcal{C}(I, X)\big); \quad u \longrightarrow \big(\Gamma \circ \mathcal{S}_G\big)(u) = \Gamma\big(\mathcal{S}_G\big)$$

Is closed graph operator in $\mathcal{C}(I, X) \times \mathcal{C}(I, X)$

Our main result may be presented as the following theorem.

Theorem 3.1: :

Assume that hypotheses $(H1) - (H6)$ hold, then the problem (1) has at least one mild solution on J_1.

Proof 3.1: :

Let $\mathbf{C} := \mathcal{C}(J_1, X)$ be the Banach space of continuous function from J_1 into X endowed with the sup-norm :

$$\|u\|_\infty := sup\{|u| : t \in [-r, b]\}; for \quad u \in \mathbf{C}$$

Transform the problem into a fixed point problem. Consider the multivalued map, $\mathcal{N} : \mathbf{C} \longrightarrow 2^{\mathbf{C}}$ defined by :

$$\mathcal{N}u := \Bigg\{ h \in \mathbf{C} :$$

$$h(t) = \begin{cases} \varphi(t); t \in J_0 \\ R(t)\mathcal{F}(0,\ \varphi) + F(t,\ u_t) \\ + \int_0^t R(t-s)g(s)\mathrm{d}s; t \in J \end{cases} \Bigg\}$$

Where : $g \in \mathcal{S}_{G,u} = \{g \in L^1(J, X) : g(t) \in G(t, u_t); \quad t \in J\}$

We have that the fixed points of N are mild solutions to (1). Now we shall prove that $\mathcal{N}$ is a completely continuous multivalued map, u.s.c, with convex closed values. The proof will be given in several steps.

Step 1 : $\mathcal{N}u$ is convex for each $u \in \mathbf{C}$.
Indeed, if h_1, h_2 belong to $\mathcal{N}u$, then there exist $g_1, g_2 \in \mathcal{S}_{G,u}$ such that for each $t \in J$ we have:

$$h_1(t) = R(t)\mathcal{F}(0,\ \varphi) + F(t,\ u_t) + \int_0^t R(t-s)g_1(s)\mathrm{d}s$$

and

$$h_2(t) = R(t)\mathcal{F}(0,\ \varphi) + F(t,\ u_t) + \int_0^t R(t-s)g_2(s)\mathrm{d}s$$

Let $0 \le k \le 1$. Then for each $t \in J$ we have :

$$\Big(kh_1 + (1-k)h_2\Big)(t) = R(t)\mathcal{F}(0,\ \varphi) + F(t,\ u_t) + \int_0^t R(t-s)\Big(kg_1(s) + (1-k)g_2(s)\Big)\mathrm{d}s$$

Thus $kh_1 + (1-k)h_2 \in \mathcal{N}u$ $\Big($ because $\mathcal{S}_{G,u}$ is convex$\Big)$, then $\mathcal{N}u$ is convex for each $u \in \mathbf{C}$

Step 2 : We will prove that $\mathcal{N}$ is a completely continuous operator. Using $(H6)$ it suffices to show that the operator $\mathcal{N}_1 : \mathbf{C} \longrightarrow 2^{\mathbf{C}}$ defined by : $\mathcal{N}_1 u := \Bigg\{ h_1 \in \mathbf{C} :$

$$h_1(t) = \begin{cases} \varphi(t); t \in J_0 \\ R(t)\mathcal{F}(0,\ \varphi) \\ + \int_0^t R(t-s)g(s)\mathrm{d}s; t \in J \end{cases} \Bigg\}$$

is completly continuous .

i- $\mathcal{N}_1$ map bounded set into bounded set in $\mathbf{C}$:
Indeed, it is enough to show that there exists a positive constant l such that for each $h_1 \in \mathcal{N}_1 u$; $u \in \mathbf{B}_q = \{u \in \mathbf{C} : \|u\|_\infty \le q\}$ we have $\|h_1\|_\infty \le l$.
If $h_1 \in \mathcal{N}_1 u$ then there exist $g \in \mathcal{S}_{G,u}$, such that for every $t \in J$ we have :

$$h_1(t) = R(t)\mathcal{F}(0,\ \varphi) + \int_0^t R(t-s)g(s)\mathrm{d}s$$

By $(H1)-(H3)$, and $(H5)$ we have for each $t \in J$:

$$\begin{aligned} |h_1(t)| &\le \|R(t)\mathcal{F}(0,\ \varphi)\| + \int_0^t \|R(t-s)g(s)\|\mathrm{d}s \\ &\le M_1[c_1M_2 + c_2] \\ &+ M_1 \sup_{u\in[0,q]} \Psi(u)\Big(\int_0^t p(s)\mathrm{d}s\Big) \end{aligned}$$

Then for each $h \in \mathcal{N}_1(B_q)$:

$$\begin{aligned} \|h_1(t)\|_\infty &\le M_1[c_1M_2 + c_2] \\ &+ M_1 \sup_{u\in[0,q]} \Psi(u)\Big(\int_0^b p(s)\mathrm{d}s\Big) \end{aligned}$$

Then $\mathcal{N}_1$ is bounded.

ii- $\mathcal{N}_1$ maps bounded set into equicontinuous sets of $\mathbf{C}$:

Let $\tau_1, \tau_2 \in J; \tau_1 < \tau_2$, and B_q be bounded set of $\mathbf{C}$; for each $u \in B_q$ and $h_1 \in \mathcal{N}_1 u$; there exist $g \in \mathcal{S}_{G,u}$ such that :

$$h_1(t) = R(t)\mathcal{F}(0,\ \varphi) + \int_0^t R(t-s)g(s)\mathrm{d}s; \quad t \in J$$

Thus,

$$\begin{aligned} h_1(\tau_2) - h_1(\tau_1) &= R(\tau_2)\mathcal{F}(0,\ \varphi) \\ &+ \int_0^{\tau_2} R(\tau_2 - s)g(s)\mathrm{d}s - R(\tau_1)\mathcal{F}(0,\ \varphi) \\ &- \int_0^{\tau_1} R(\tau_1 - s)g(s)\mathrm{d}s \\ &= \Big(R(\tau_2) - R(\tau_1)\Big)\mathcal{F}(0,\ \varphi) \\ &+ \int_0^{\tau_1} \Big(R(\tau_2 - s) - R(\tau_1 - s)\Big)g(s)\mathrm{d}s \\ &+ \int_{\tau_1}^{\tau_2} R(\tau_2 - s)g(s)\mathrm{d}s \end{aligned}$$

Then

$$\begin{aligned} \|h_1(\tau_2) - h_1(\tau_1)\| &\le \|R(\tau_2) - R(\tau_1)\| \\ &+ \int_0^{\tau_1} \|R(\tau_2 - s) \\ &- R(\tau_1 - s)\|\|g(s)\|ds \\ &+ \int_{\tau_1}^{\tau_2} \|R(\tau_2 - s)\|\|g(s)\|ds \end{aligned}$$

As $\tau_2 \longrightarrow \tau_1$ the right-hand side of the above inequality tends to zero, implies that $\mathcal{N}_1 u$ is equicontinuous on J_1

iii- $V(t) = \{h_1(t); h_1 \in \mathcal{N}_1(B_q)\}$ is relatively compact on X:
By $(H4)$ $V(t)$ is relatively compact for $t = 0$; let $0 \le t \le b$ be fixed and let ε be a real number satisfying $0 \le \varepsilon < t$ for $u \in B_q$ and $g \in \mathcal{S}_{G,u}$ such that :

$$h_1(t) = R(t)\mathcal{F}(0,\ \varphi) + \int_0^t R(t-s)g(s)\mathrm{d}s; \quad t \in J$$

and,

$$h_{1,\varepsilon}(t) = R(t)\mathcal{F}(0,\ \varphi) + \int_0^{t-\varepsilon} R(t-s)g(s)\mathrm{d}s; \quad t \in J$$

The set $V_\varepsilon(t) = \{h_{1,\varepsilon}(t); h_{1,\varepsilon} \in \mathcal{N}_1(B_q)\}$ is relatively compact beceuse $R(t)$ is compact then;

$$\begin{aligned}\|h_1(t) - h_{1,\varepsilon}(t)\| &= \int_{t-\varepsilon}^{t} R(t-s)g(s)\| \\ &\leq M_1 \sup_{u\in[0,q]} \Psi(u) \int_0^t p(s)\mathrm{d}s.\varepsilon \\ &\leq r\varepsilon\end{aligned}$$

With; $r = M_1 \sup_{u\in[0,q]} \Psi(u) \int_0^b p(s)ds$, this implies that $V(t)$ is relativement compact thus by $(i),(ii),(iii)$ and by Arzela-Ascoli theorem, we can deduce that $\mathcal{N}_1$ is completely continuous then $\mathcal{N} : \mathbf{C} \longrightarrow 2^{\mathbf{C}}$ is completely continuous.

Step 3 : $\mathcal{N}$ has a closed graph.
Let $u_n \longrightarrow u$, $h_n \in \mathcal{N}u_n$ and $h_n \longrightarrow h$, we shall prove that $h \in \mathcal{N}u$.
$h_n \in \mathcal{N}u_n$ then there exists $g_n \in \mathcal{S}_{G,u_n}$ such that

$$\begin{aligned}h_n(t) &= R(t)\mathcal{F}(0,\ \varphi) + F(t, u_{nt}) \\ &+ \int_0^t R(t-s)g_n(s)\mathrm{d}s; \quad t \in J\end{aligned}$$

We should prove that $g \in \mathcal{S}_{G,u}$ such that for each $t \in J$:

$$\begin{aligned}h(t) &= R(t)\mathcal{F}(0,\ \varphi) + F(t, u_t) \\ &+ \int_0^t R(t-s)g(s)\mathrm{d}s; \quad t \in J\end{aligned}$$

Since F is continuous, we have that:

$$\begin{aligned}&\|\Big(h_n(t) - R(t)\mathcal{F}(0,\ \varphi) - F(t, u_{nt})\Big) \\ &- \Big(h(t) - R(t)\mathcal{F}(0,\ \varphi) - F(t, u_t)\Big)\|_\infty \longrightarrow 0\end{aligned}$$

As $n \longrightarrow \infty$.
Consider the linear operator :

$$\Gamma : \mathbf{L}^1(J, X) \longrightarrow \mathcal{C}(J, X)$$

$$g \longrightarrow \Gamma(g)(t) = \int_0^t R(t-s)g(s)ds$$

From Lemma 3.1; $\Gamma \circ \mathcal{S}_G$ is a closed graph operator then we have that :

$$h_n(t) - R(t)\mathcal{F}(0,\ \varphi) - F(t, u_{nt}) \in \Gamma\Big(\mathcal{S}_{G,u_n}\Big)$$

Since $u_n \longrightarrow u$, and by the lemma 3.1 :

$$h(t) - R(t)\mathcal{F}(0,\ \varphi) - F(t, u_t) \in \Gamma\Big(\mathcal{S}_{G,u}\Big)$$

It follows that $g \in \mathcal{S}_{G,u}$ such that

$$\begin{aligned}h(t) &= R(t)\mathcal{F}(0,\ \varphi) + F(t, u_t) \\ &+ \int_0^t R(t-s)g(s)\mathrm{d}s; t \in J\end{aligned}$$

From the Step 1, step 2 and step 3 we deduce that $\mathcal{N}$ is u.s.c, completely continuous then by lemma (2.3), $\mathcal{N}$ is a condensing, bounded, closed and convex operator. In order to prove that $\mathcal{N}$ has a fixed point, we need one more step.

Step 4 : The set $\Omega := \{u \in X : \lambda u \in \mathcal{N}u \quad \text{for} \quad \lambda > 1\}$ is bounded. Let $u \in \Omega$. Then $\lambda u \in \mathcal{N}u$, thus there exists $g \in \mathcal{S}_{G,u}$ such that :

$$\begin{aligned}u(t) &= \lambda^{-1}R(t)\mathcal{F}(0,\ \varphi) + \lambda^{-1}F(t, u_t) \\ &+ \lambda^{-1}\int_0^t R(t-s)g(s)\mathrm{d}s\end{aligned}$$

By $(H_1) - (H_3)$and (H_5) we have :

$$\begin{aligned}|u(t)| &\leq M_1\Big((1+c_1)M_2 + c_2\Big) + c_1\|u_t\| + c_2 \\ &+ M_1\int_0^t p(s)\Psi(\|u_s\|)ds\end{aligned}$$

Consider the function defined by :

$$\mu(t) = sup\{|u(s)| : -r \leq s \leq t\}; 0 \leq t \leq b$$

Let $t^* \in [-r, t]$ be such that $\mu(t) = |u(t^*)|$.
$\star$ If $t^* \in J_0 = [-r, 0]$ then :

$$\mu(t) \leq \|\phi\| \leq M_2$$

$\star\star$ If $t^* \in J = [0, b]$ then :

$$\begin{aligned}\mu(t) &\leq M_1\Big((1+c_1)M_2 + c_2\Big) + c_1\mu(t) + c_2 \\ &+ M_1\int_0^t p(s)\Psi\big(\mu(s)\big)ds\end{aligned}$$

then;

$$\begin{aligned}\mu(t) &\leq \frac{1}{1-c_1}\Bigg(M_1\Big((1+c_1)M_2 + c_2\Big) + c_2 \\ &+ M_1\int_0^t p(s)\Psi\big(\mu(s)\big)ds\Bigg)\end{aligned}$$

since $M_1 \geq 1$ Let us take the right-hand side of the above inequality as $\nu(t)$. Then we have.
$c = \nu(0) = \dfrac{1}{1-c_1}\Bigg(M_1\big(M_2(1+c_1) + c_2\big) + c_2\Bigg)$ and $\mu(t) \leq \nu(t) \quad ; \quad \forall t \in J$ then,

$$\nu'(t) = \frac{1}{1-c_1}M_1 p(t)\Psi\big(\mu(t)\big)$$

By using $H(5)$ we get :

$$\nu' < \omega(t)\Psi\big(\nu(t)\big)$$

This implies that

$$\int_{\nu(0)=c}^{\nu(t)} \frac{d\tau}{\Psi(\tau)} < \int_{0}^{b} \omega(s)ds < \int_{c}^{\infty} \frac{d\tau}{\tau+\Psi(\tau)}$$

This implies that there exists a constant K such that $\nu \leq K$, $t \in J$ and $\mu \leq K$, $t \in J$. Since for every $t \in J$ we have $\|u_t\| \leq \mu(t)$ then

$$\|u\|_{\infty} := sup\{|u(t)|; -r \leq t \leq b\} \leq K$$

Where K depends only on b and on the functions p and Ψ. This shows that Ω is bounded.
As a consequence of theorem 2.4 (Leray-schauders's fixed point) we deduce that $\mathcal{N}$ has a fixed point which is a solution of (1).

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