Fuzzy fractional differential wave equation

Said Melliani *, M'hamed Elomari *, Khalid Hilal * & Manal Menchih *

* Laboratory of Applied Mathematics and Scientific Computing, Sultan Moulay Slimane University, Morocco

Email: said.melliani@gmail.com, m.elomari@usms.ma, {hilalkhalid2005, manalmenchih97}@yahoo.fr

Abstract—**This work is related to investigate integral solution of wave equation with fuzzy initial data under generalized fuzzy Caputo derivative. For the concerned investigation, we use the Fourier transform. The exact solution is given in the case of $\gamma = 2$. Some examples are presented to illustrate the results.**

Index Terms—**Generalized fuzzy derivative, Caputo fractional derivative, Hukuhara difference, fuzzy fourier transform.**

I. Introduction

The present paper investigate the analytic solution of the following problem

$$\begin{cases} {}_{gH}D_t^{\gamma} u(t,x) -_g c^2 \frac{\partial^2}{\partial x^2} u(t,x) = 0, \\ -\infty < x < \infty,\ t \geq 0,\ 1 < \gamma < 2 \\ u(0,x) = a(x) \\ \frac{\partial}{\partial t} u(0,x) = b(x) \end{cases}$$

where a and b are two absolutely valued-functions in E^1. $-g$ is the generalized Hukuhara difference. ${}_{gH}D$ is the generalized fuzzy fractional caputo's derivative.

In 1965 L.Zadeh [13] introduced the basic ideas of the fuzzy set theory, as an extension of the classical notion of set. The authors in [6] give a generalization of the Hukuhara difference which guaranteed the existence of this is for two segments in $\mathbb{R}$. As consequance in the same work Bede and Stefanini presented the generalized derivative of a set valued-functions. Agarwal et al. [1] are the pioneers working in fuzzy fractional (DEs). They formulated the Riemann-Liouville differentiability notion as the base to define the concept of fuzzy fractional DEs. After that, they proved the existence of solutions of fuzzy fractional integral equations (IEs) under compactness type conditions using the Hausdorff measure of non-compactness in the paper [2]. Allahviranloo et all in [3] presented two new results on the existence of two kinds of $gH-$weak solutions of these problems and indicated the boundedness and continuous dependence of solutions on the initial data of the problems. In [5] the authors prove the existence and uniqueness theorems for non-linear fuzzy fractional Fredholm integro-differential equations under fractional generalized Hukuhara derivatives in the Caputo sense. From the idea of [5] we will try to prove the existence and uniqueness of fuzzy fractional wave equation.

This paper is organized as follows. In section 2 we recall some concepts concerning the fuzzy metric space. the generalized derivative take place in the section 3. In section 4 we give the concept of fuzzy Fourier transform and we presented some properties. We presented the solution of the fuzzy wave equation in section 5. Finally in section 6 two examples are given to illustrate the usefulness of our main results.

II. preliminaries

In this section, we present some definitions and introduce the necessary notation, which will be used throughout the paper.

We denote E^1 the class of function defined as follows:

$$E^1 = \left\{ u : \mathbb{R} \to [0,1], \quad u \quad \text{satisfies } (1-4) \text{ below} \right\}$$

1) u is normal, i.e. there is a $x_0 \in \mathbb{R}$ such that $u(x_0) = 1$;
2) u is a fuzzy convex set;
3) u is upper semi-continuous;
4) u closure of $\{x \in \mathbb{R}^n, \quad u(x) > 0\}$ is compact

For all $\alpha \in (0,1]$ the α-cut of an element of E^1 is defined by

$$u^\alpha = \left\{ x \in \mathbb{R}, \; u(x) \geq \alpha \right\}$$

By the previous properties we can write

$$u^\alpha = [\underline{u}(\alpha), \overline{u}(\alpha)]$$

By the extension principal of Zadeh we have

$$\begin{aligned} (u+v)^\alpha &= u^\alpha + v^\alpha; \\ (\lambda u)^\alpha &= \lambda u^\alpha \end{aligned}$$

For all $u, v \in E^1$ and $\lambda \in \mathbb{R}$

The distance between two element of E^1 is given by (see [4])

$$d(u,v) = \sup_{\alpha \in (0,1]} \max \left\{ |\overline{u}(\alpha) - \overline{v}(\alpha)|, |\underline{u}(\alpha) - \underline{v}(\alpha)| \right\}$$

The metric space (E^1, d) is complete, separable and locally compact and the following properties for metric d are valid:

1) $d(u+v, u+w) = d(u,v)$;
2) $d(\lambda u, \lambda v) = |\lambda| d(u,v)$;
3) $d(u+v, w+z) \leq d(u,w) + d(v,z)$;

Remark II.1 *The space (E^1, d) is a linear normed space with $\|u\| = d(u,0)$.*

Definition II.2 *[10] A complex fuzzy number is a mapping $z : \mathbb{C} \to [0,1]$ with the following properties:*

1) *z is continuous;*
2) *z^α, $\alpha \in (0,1]$ is open, bounded, connected and simply connected;*
3) *z^1 is non-empty, compact, arcwise connected and simply connected.*

We denote the set of all fuzzy complex number by $\mathbb{C}^1$.

Definition II.3 *[6] The generalized Hukuhara difference of two fuzzy numbers $u, v \in E^1$ is defined as follows*

$$u -_g v = w \Leftrightarrow \begin{cases} u = v + w \\ or \quad v = u + (-1)w \end{cases}$$

In terms of α-levels we have

$$\left(u -_g v\right)^\alpha = \Big[\min\left\{\underline{u}(\alpha) - \underline{v}(\alpha), \overline{u}(\alpha) - \overline{v}(\alpha)\right\}, \max\left\{\underline{u}(\alpha) - \underline{v}(\alpha), \overline{u}(\alpha) - \overline{v}(\alpha)\right\}\Big]$$

and the conditions for the existence of $w = u -_g v \in E^1$ are

$$\text{case (i)} \quad \begin{cases} \underline{w}(\alpha) = \underline{u}(\alpha) - \underline{v}(\alpha) \text{ and } \overline{w}(\alpha) = \overline{u}(\alpha) - \overline{v}(\alpha) \\ \text{with} \quad \underline{w}(\alpha) \text{ increasing,} \\ \overline{w}(\alpha) \text{ decreasing}, \underline{w}(\alpha) \leq \overline{w}(\alpha) \end{cases}$$

$$\text{case (ii)} \quad \begin{cases} \underline{w}(\alpha) = \overline{u}(\alpha) - \overline{v}(\alpha) \text{ and } \overline{w}(\alpha) = \underline{u}(\alpha) - \underline{v}(\alpha) \\ \text{with} \quad \underline{w}(\alpha) \text{ increasing,} \\ \overline{w}(\alpha) \text{ decreasing}, \underline{w}(\alpha) \leq \overline{w}(\alpha) \end{cases}$$

for all $\alpha \in [0,1]$.

Throughout the rest of this paper, we assume that $u -_g v \in E^1$

Proposition II.4 *[11]*

$$\|u -_g v\| = d(u,v)$$

Since $\|.\|$ is a norm on E^n and by the proposition (II.4) we have

Proposition II.5

$$\|\lambda u -_g \mu u, 0\| = |\lambda - \mu| \|u\|$$

Let $f : [a,b] \subset \mathbb{R} \to E^1$ a fuzzy-valued function. The α-level of f is given by

$$f(x,\alpha) = \left[\underline{f}(x,\alpha), \overline{f}(x,\alpha)\right], \; \forall x \in [a,b], \; \forall \alpha \in [0,1].$$

Definition II.6 *[6] Let $x_0 \in (a,b)$ and h be such that $x_0+h \in (a,b)$, then the generalized Hukuhara derivative of a fuzzy value function $f:(a,b) \to E^1$ at x_0 is defined as*

$$\lim_{h\to 0}\left\|\frac{f(x_0+h) -_g f(x_0)}{h} -_g f'_{gH}(x_0)\right\| = 0 \qquad \text{(II.1)}$$

If $f_{gH}(x_0) \in E^1$ satisfying (II.1) exists, we say that f is generalized Hukuhara differentiable (gH-differentiable for short) at x_0.

Definition II.7 *[6] Let $f:[a,b] \to E^1$ and $x_0 \in (a,b)$, with $\underline{f}(x,\alpha)$ and $\overline{f}(x,\alpha)$ both differentiable at x_0. We say that*

1) *f is $[(i)-gH]$-differentiable at x_0 if*

$$f'_{i,gH}(x_0) = \left[\underline{f}'(x,\alpha), \overline{f}'(x,\alpha)\right] \qquad \text{(II.2)}$$

2) *f is $[(ii)-gH]$-differentiable at x_0 if*

$$f'_{ii,gH}(x_0) = \left[\overline{f}'(x,\alpha), \underline{f}'(x,\alpha)\right] \qquad \text{(II.3)}$$

Theorem II.8 *Let $f: J \subset \mathbb{R} \to E^1$ and $g: J \to \mathbb{R}$ and $x \in J$. Suppose that $g(x)$ is differentiable function at x and the fuzzy-valued function $f(x)$ is gH-differentiable at x. So*

$$(fg)'_{gH} = (f'g)_{gH} + (fg')_{gH}$$

Proof *Using (II.5), for h enough small we get*

$$\begin{aligned}
&\left\|\frac{f(x+h)g(x+h) -_g f(x)g(x)}{h} -_g \left((f'(x)g(x))_{gH} + (f(x)g'(x))_{gH}\right)\right\| \\
&= \left\|\frac{f(x+h)g(x+h) -_g f(x)g(x+h) + f(x)g(x+h) -_g f(x)g(x)}{h} -_g \left((f'(x)g(x))_{gH} + (f(x)g'(x))_{gH}\right)\right\| \\
&= \left\|\frac{(f(x+h) -_g f(x))g(x+h) + f(x)(g(x+h) -_g g(x))}{h} -_g \left((f'(x)g(x))_{gH} + (f(x)g'(x))_{gH}\right)\right\| \\
&\leq \left\|\frac{(f(x+h) -_g f(x))g(x+h)}{h} -_g \left((f'(x)g(x))_{gH}\right)\right\| + \left\|\frac{(f(x)(g(x+h) -_g g(x))}{h} -_g \left((f(x)g'(x))_{gH}\right)\right\| \\
&\leq \left\|\frac{(f(x+h) -_g f(x))}{h} g(x+h) -_g \left((f'(x)g(x))_{gH}\right)\right\| + \left\|f(x)\frac{((g(x+h) -_g g(x))}{h} -_g \left((f(x)g'(x))_{gH}\right)\right\|
\end{aligned}$$

which complet the proof by passing to limit.

Definition II.10 *[6] We say that a point $x_0 \in (a,b)$, is a switching point for the differentiability of f, if in any neighborhood V of x_0 there exist points $x_1 < x_0 < x_2$ such that*

1) *type (1). at x_1 (II.2) holds while (II.3) does not hold and at x_2 (II.3) holds and (II.2) does not hold, or*
2) *type (2). at x_1 (II.3) holds while (II.2) does not hold and at x_2 (II.2) holds and (II.3) does not hold.*

Definition II.11 *[3] Let $f:(a,b) \to E^1$. We say that $f(x)$ is gH-differentiable of the 2nd-order at x_0 whenever the function $f(x)$ is gH-differentiable of the order $i, i = 0,1$, at x_0, $((f(x_0))^{(i)}_{gH} \in E^1)$, moreover there isn't any switching point on (a,b). Then there exists $(f)''_{gH}(x_0) \in E^1$ such that*

$$\lim_{h\to 0}\left\|\frac{f'(x_0+h) -_g f'(x_0)}{h}, f''_{gH}(x_0)\right\| = 0$$

Definition II.12 *[3] Let $f:[a,b] \to E^1$ and $f'_g H(x)$ be gH-differentiable at $x_0 \in (a,b)$, moreover there isn't any switching point on (a,b) and $\underline{f}(x,\alpha)$ and $\overline{f}(x,\alpha)$ both differentiable at x_0. We say that*

- *f' is $[(i)-gH]$-differentiable at x_0 if*

$$f''_{i,gH}(x_0) = \left[\underline{f}''(x,\alpha), \overline{f}''(x,\alpha)\right]$$

- *f' is $[(ii)-gH]$-differentiable at x_0 if*

$$f''_{ii,gH}(x_0) = \left[\overline{f}''(x,\alpha), \underline{f}''(x,\alpha)\right]$$

Definition II.13 *[8] Let $f:[a,b] \to E^1$. We say that $f(x)$ is fuzzy Riemann integrable to $I \in E^1$ if for any $\epsilon > 0$, there exists $\delta > 0$ such that for any division $P = \{[u,v];\xi\}$ with the norms $\Delta(P) < \delta$, we have*

$$d\left(\sum_p^{*}(v-u)f(\xi), I\right) < \epsilon$$

where $\sum_p^*$ denotes the fuzzy summation. We choose to write $I = \int_a^b f(x)dx$.

Theorem II.14 *[6] If f is gH-differentiable with no switching point in the interval $[a,b]$ then we have*

$$\int_a^b f(t)dt = f(b) -_g f(a)$$

Theorem II.15 *[12] Let $f(x)$ be a fuzzy-valued function on $(-\infty,\infty)$ and it is represented by $f(x,\alpha) = \left[\underline{f}(x,\alpha), \overline{f}(x,\alpha)\right]$for any fixed $\alpha \in [0,1]$. Assume that $|\underline{f}(x,\alpha)|$ and $|\overline{f}(x,\alpha)|$ are Riemann integrable on*

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$(-\infty,\infty)$ *for all* $\alpha \in [0,1]$. *Then* $f(x)$ *is improper fuzzy Riemann-integrable on* $(-\infty,\infty)$ *and the improper fuzzy Riemann integral is a fuzzy number. Furthermore, we have*

$$\int_{-\infty}^{\infty} f(x)dx = \left[\int_{-\infty}^{\infty} \underline{f}(x,\alpha)dx, \int_{-\infty}^{\infty} \overline{f}(x,\alpha)dx\right]$$

From this theorem we can discuss the Fuzzy Riemann's improper integral

Lemma II.16 *Let* $f : \mathbb{R} \times \mathbb{R}^{+} \rightarrow E^1$, *given by* $f(x,t;\alpha) = [\underline{f}(x,t;\alpha), \overline{f}(x,t;\alpha)]$, *and let* $a \in \mathbb{R}^{+}$
If $\int_a^{\infty} \underline{f}(x,t;\alpha)dt$ *and* $\int_a^{\infty} \overline{f}(x,t;\alpha)dt$ *are converges then*

$$\int_a^{\infty} f(x,t;\alpha)dt \in E^1$$

Proof *Just use the conditions* (II.1).

Theorem II.18 *Let* $f : \mathbb{R} \times \mathbb{R}^{+} \rightarrow E^1$ *be fuzzy-valued function such that* $f(x,t;\alpha) = [\underline{f}(x,t;\alpha), \overline{f}(x,t;\alpha)]$. *Suppose that for each* $x \in [a,\infty)$, *the fuzzy integral* $\int_c^{\infty} f(x,t)dt$ *is convergent and moreover* $\int_a^{\infty} f(x,t)dx$ *as a function of* t *is convergent on* $[c,\infty)$. *Then*

$$\int_c^{\infty}\int_a^{\infty} f(x,t)dxdt = \int_a^{\infty}\int_a^{\infty} f(x,t)dtdx$$

Proof *Applying the theorem of Fubini-Tonelli [7] to these two functions* $\underline{f}(x,t;\alpha)$ *and* $\overline{f}(x,t;\alpha)$, *and use the conditions* (II.1)

Theorem II.20 *Suppose both,* $f(x,t)$ *and* $\partial_{x_{gH}} f(x,t)$, *are fuzzy continuous in* $[a,b] \times [c,\infty)$. *Suppose also that the integral converges for* $x \in \mathbb{R}$, *and the integral* $\int_c^{\infty} f(x,t)dt$ *converges uniformly on* $[a,b]$. *Then* F *is gH-differentiable on* $[a,b]$ *and*

$$F'_{gH}(x) = \int_c^{\infty} \partial_{x_{gH}} f(x,t)dt$$

Proof *The continuity of* $\partial_{x_{gH}} f(x,t)$ *on* $[a,b]$ *by the convergence domainee theorem of to* $\underline{f}(x,t;\alpha)$ *and* $\overline{f}(x,t;\alpha)$ *and use the condition* (II.1).

According to the theorem (II.8) we get

Theorem II.22 *Let* $f : [a,b] \rightarrow E^1$ *and* $g : [a,b] \rightarrow \mathbb{R}$ *are two differentiable functions (*f *is gH-differentiable), then*

$$\int_a^b f'_{gH}(x)g(x)dx = f(b)g(b) -_g f(a)g(a) -_g \int_a^b f(x)g'(x)dx$$

Remark II.23 *If* $f,g \in A^{E^1}$ *with* $\lim_{|x|\rightarrow\infty} f(x) = 0$, $\lim_{|x|\rightarrow\infty} g(x) = 0$ *then*

$$\int_{-\infty}^{\infty} f'_{gH}(x)g(x)dx = \int_{-\infty}^{\infty} f(x)g'(x)dx$$

III. Fuzzy generalized Hukuhara partial differentiation

In this section $f : \mathbb{D} \subset \mathbb{R} \times \mathbb{R}^{+} \rightarrow E^1$ is called the two variable fuzzy-valued function. The parametric representation of the fuzzy-valued function fis expressed by $f(x,t,\alpha) = \left[\underline{f}(x,t,\alpha), \overline{f}(x,t,\alpha)\right]$

Definition III.1 *[3] Let* $f : \mathbb{D} \subset \mathbb{R} \times \mathbb{R}^{+} \rightarrow E^1$ *and* $(x_0,t_0) \in \mathbb{D}$. *Then first generalized Hukuhara partial derivative (*$[gH-p]$*-derivative for short) of* f *with respect to variables* x,t *are the functions* $\partial_{x_{gH}} f(x_0,t_0)$ *and* $\partial_{t_{gH}} f(x_0,t_0)$ *given by*

$$\lim_{h\rightarrow 0}\left\|\frac{f(x_0+h,t_0) -_g f(x_0,t_0)}{h} -_g \partial_{x_{gH}} f(x_0,t_0)\right\| = 0$$

and

$$\lim_{h\rightarrow 0}\left\|\frac{f(x_0,t_0+h) -_g f(x_0,t_0)}{h}, \partial_{x_{gH}} f(x_0,t_0)\right\| = 0$$

provided that $\partial_{x_{gH}} f(x_0,t_0), \partial_{t_{gH}} f(x_0,t_0) \in E^1$.

Definition III.2 *[3] Let* $f(x,t) : \mathbb{D} \rightarrow E^1$, $(x_0,t_0) \in \mathbb{D}$ *and* $\underline{f}(x,t;\alpha)$ *and* $\overline{f}(x,t;\alpha)$ *both partial differentiable w.r.t.* t *at* (x_0,t_0). *We say that*

- $f(x,t)$ *is* $[(i)-p]$*-differentiable w.r.t.* t *at* (x_0,t_0) *if*

$$\partial_{t_{i,gH}} f(x_0,t_0) = \left[\partial_t \underline{f}(x_0,t_0;\alpha), \partial_t \overline{f}(x_0,t_0;\alpha)\right] \quad \text{(III.1)}$$

$$\partial_{t_{ii,gH}} f(x_0,t_0) = \left[\partial_t \overline{f}(x_0,t_0;\alpha), \partial_t \underline{f}(x_0,t_0;\alpha)\right] \quad \text{(III.2)}$$

We inspired of the definition (II.11) we presented the following definition

Definition III.3 *$f : \mathbb{R} \times \mathbb{R}^+ R \to E^1$. We say that the function $t = h(x)$, is switching boundary for the differentiability of $f(x,t)$ with respect to t, if for all x belongs to domain of $h(x)$ and for all $t \in \mathbb{R}^+$, there exist points $t_0 < t_1 < t_2$ such that*

1) *at (x,t_1) (III.1) holds while (III.2) does not hold and at (x,t_2) (III.2) holds and (III.1) does not hold, or*
2) *at (x,t_1) (III.2) holds while (III.1) does not hold and at (x,t_2) (III.1) holds and (III.2) does not hold.*

Theorem III.4 *Consider $f : \mathbb{R} \times \mathbb{R}^+ \to E^1$ and $u : \mathbb{R} \to E^1$ are fuzzy-valued functions such that $u(x;\alpha) = [\underline{u}(x;\alpha), \overline{u}(x;\alpha)]$. Suppose that $h : \mathbb{R} \to \mathbb{R}$and $p : \mathbb{R} \times \mathbb{R}^+ \to \mathbb{R}^+$ is a differentiable function w.r.t. t and*

$$\partial_t p(x,t) = \begin{cases} \partial_t p(x,t) \geq 0, & h_1(t) < x < h_2(t); \\ \partial_t p(x,t) < 0, & h_2(t) < x < h_3(t) \end{cases}$$

and $f(x,t) = p(x,t)u(x)$. Then $\partial_{t_{gH}} f(x,t)$ exists and

$$\partial_{t_{gH}} p(x,t) = \begin{cases} \partial_{t_{i,gH}} p(x,t) \geq 0, & h_1(t) < x < h_2(t); \\ \partial_{t_{ii,gH}} p(x,t) < 0, & h_2(t) < x < h_3(t) \end{cases}$$

In fact, the function $h_2(t)$ is switching boundary type 1 for differentiability of $f(x,t)$ with respect to t.

Proof *Since p is valued in $\mathbb{R}^+$ then we can set $f(x,t;\alpha) = p(x,t)[\underline{u}(x;\alpha), \overline{u}(x;\alpha)]$, which implies that*

$$\partial_{t_{gH}} = \partial_t p(x,t)[\underline{u}(x;\alpha), \overline{u}(x;\alpha)]$$

.

If $h_1(t) < x < h_2(t)$ then

$$\partial_{t_{gH}} = [\partial_t p(x,t)\underline{u}(x;\alpha), \partial_t p(x,t)\overline{u}(x;\alpha)]$$

then $f(x,t)$ is [(i)-differentiable] by report at t. In the same if $h_2(t) < x < h_3(t)$ we get

$$\partial_{t_{gH}} = [\partial_t p(x,t)\overline{u}(x;\alpha), \partial_t p(x,t)\underline{u}(x;\alpha)]$$

thus $f(x,t)$ is [(ii)-differentiable] by report at t

IV. Generalized fuzzy fractional derivative

We present generalized fuzzy fractional derivative and their properties.

Definition IV.1 *[5] Let $f \in A^{E^1}([a,b])$. The fuzzy Riemann-Liouville integral of fuzzy-valued function f is defined as following:*

$$I^q f(t) = \frac{1}{\Gamma(1-q)} \int_a^t (t-s)^{q-1} f(s) ds,$$
$$a < s < t, \quad 0 < q < 1.$$

Definition IV.2 *[5] Let $f(x,t;\alpha) = [\underline{f}(x,t;\alpha), \overline{f}(x,t;\alpha)]$ be a valued-fuzzy function. The fuzzy Riemann-Liouville integral of f is defined as following:*

$${}_{gH}D_t^q f(t,x;\alpha) = \frac{1}{\Gamma(1-q)} \int_a^t (t-s)^q f'_{gH}(s) ds,$$
$$a < s < t, \quad 0 < q < 1$$

Also we say that f is $[(i) - gH]$-differentiable at t_0 if

$${}_{gH}D_t^q f(x,t;\alpha) = [D^q \underline{f}(x,t;\alpha), \overline{f}(x,t;\alpha)]$$

and f is $[(ii) - gH]$-differentiable at t_0 if

$${}_{gH}D_t^q f(x,t;\alpha) = [D^q \overline{f}(x,t;\alpha), \underline{f}(x,t;\alpha)]$$

Lemma IV.3 *Let $f \in A^{E^1}$ and $r \in (0,1)$,then*

1) *If f is $[(i) - gH]$-differentiable at t_0 then $D^r f$ is $[(i) - gH]$-differentiable at t_0.*
2) *If f is $[(ii) - gH]$-differentiable at t_0 then $D^r f$ is $[(ii) - gH]$-differentiable at t_0*

Proof *Note that*

$${}_{gH}D^q f(t) = \frac{1}{\Gamma(1-q)} \int_0^t (t-s)^{-q} f'_{gH}(s) ds$$

Since $\frac{1}{\Gamma(1-q)}(t-s)^{-q}$ is a nonegative quantity whenever $0 < t < s$.

Theorem IV.5 *Let $f \in A^{E^1}$ and $q \in (1,2)$,then*

$${}_{gH}D^q f(t) = {}_{gH}D^{q-1} f'_{gH}(t)$$

Proof *We set $f(t) = [\underline{f}(t;\alpha), \overline{f}(t;\alpha)]$and use lemma (IV.3)*
If f is [(i)-differentiable] then

$$f(t)' = [\underline{f}'(t;\alpha), \overline{f}'(t;\alpha)]$$

and

$$D^{q-1}f(t)' = [D^{q-1}\underline{f}'(t;\alpha), D^{q-1}\overline{f}'(t;\alpha)]$$

If f is [(i)-differentiable] then

$$f(t)' = [\overline{f}'(t;\alpha), \underline{f}'(t;\alpha)]$$

and

$$D^{q-1}f(t)' = [D^{q-1}\overline{f}'(t;\alpha), D^{q-1}\underline{f}'(t;\alpha)]$$

Proposition IV.7 *Let $f : L^{E^1}$.*
If $D^{\gamma-1}f(t) = g(t)$, then $f(t) = f(0) + tf'_{gH}(0) + I^{\gamma-1}g(t)$

Proof *We set $f(t) = [\underline{f}(t;\alpha), \overline{f}(t;\alpha)]$ and*
$g(t) = [\underline{g}(t;\alpha), \overline{g}(t;\alpha)]$.

1) *If f is [(i)-differentiable] by theorem (IV.5)*

$$\begin{aligned} D^{\gamma-1}f(t) &= [D^{\gamma-1}\underline{f}(t;\alpha), D^{\gamma-1}\overline{f}(t;\alpha)] \\ &= [\underline{g}(t;\alpha), \overline{g}(t;\alpha)] \end{aligned}$$

Which implies that

$$\begin{cases} D^{\gamma-1}\underline{f}(t;\alpha) = \underline{g}(t;\alpha) \\ D^{\gamma-1}\overline{f}(t;\alpha) = \overline{g}(t;\alpha) \end{cases}$$

By [9] we get

$$\begin{cases} \underline{f}(t;\alpha) = \underline{f}(0;\alpha) + t\underline{f}'(0;\alpha) + I^{\gamma-1}\underline{g}(t;\alpha) \\ \overline{f}(t;\alpha) = \overline{f}(0;\alpha) + t\overline{f}'(0;\alpha) + I^{\gamma-1}\overline{g}(t;\alpha) \end{cases}$$

in the same if f is [(ii)-differentiable] then

$$\begin{cases} \underline{f}(t;\alpha) = \underline{f}(0;\alpha) + t\overline{f}'(0;\alpha) + I^{\gamma-1}\underline{g}(t;\alpha) \\ \overline{f}(t;\alpha) = \overline{f}(0;\alpha) + t\underline{f}'(0;\alpha) + I^{\gamma-1}\overline{g}(t;\alpha) \end{cases}$$

Thus

$$f(t) = f(0) + tf'_{gH}(0) + I^{\gamma-1}g(t)$$

V. Fuzzy Fourier transform

In this section we discuss the Fourier transform in the fuzzy case

Lemma V.1 *If $f \in A^{E^1}$ then the map*

$$\begin{aligned} F : \quad & \mathbb{R} \longmapsto \mathbb{C}^1 \\ & \omega \to \textstyle\int_{-\infty}^{\infty} f(x)e^{-i\omega w}dx \end{aligned}$$

is well defined

Proof *We have*

$$\left\| f(x)e^{-i\omega w} \right\| = \left\| f(x) \right\|$$

Since $f \in A^{E^1}$ then $f(x)e^{-i\omega w} \in A^{\mathbb{C}^1}$, which complet the proof.

Remark V.3 *In the same the map and under same assumption*

$$\begin{aligned} F : \quad & \mathbb{R} \longmapsto \mathbb{C}^1 \\ & \omega \to \textstyle\int_{-\infty}^{\infty} f(x)e^{i\omega w}dx \end{aligned}$$

is well defined

By the previous lemma and remark we can give a definition of the fuzzy Fourier transform

Definition V.4 *Let $f : \mathbb{R} \to E^1$ a fuzzy-valued function. The fuzzy Fourier transform of f, denote $\mathcal{F}(f) : \mathbb{R} \to \mathbb{C}^1$, is given by*

$$\mathcal{F}(f(x)) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)e^{-i\omega w}dx = F(\omega)$$

Also the fuzzy inverse Fourier transform of $F(\omega)$ is given by

$$\mathcal{F}^{-1}(F(\omega)) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)e^{i\omega w}dx = f(x)$$

By the conditions (II.1) we have

Remark V.5 *Let $f \in A^{\mathbb{C}^1}$.*
If $f(x,t;\alpha) = [\underline{f}(x,t;\alpha), \overline{f}(x,t;\alpha)]$, then we can denote

$$\mathcal{F}(f(x,t;\alpha)) = \left[\mathcal{F}\left(\underline{f}(x,t;\alpha)\right), \mathcal{F}\left(\overline{f}(x,t;\alpha)\right)\right]$$

with

$$[z_1, z_2] = [Re(z_1), Re(z_2)] \times [Im(z_1), Im(z_2)]$$

and

$$\mathcal{F}^{-1}(f(x,t;\alpha)) = \left[\mathcal{F}^{-1}\left(\underline{f}(x,t;\alpha)\right), \mathcal{F}^{-1}\left(\overline{f}(x,t;\alpha)\right)\right]$$

Using the conditions (II.1) and the linearity of Fourier transform on a "crisp" function we get for all $a, b > 0$

$$a\mathcal{F}(f(x,t;\alpha)) + b\mathcal{F}(g(x,t;\alpha)) = \mathcal{F}(af(x,t;\alpha) + bg(x,t;\alpha))$$

Theorem V.6 *Let $f \in A^{E^1}$ such that $\lim_{|x|\to\infty} f(x) = 0$. suppose that $f'_{gH} \in A^{E^1}$. Then*

$$\mathcal{F}\left(f'_{gH}(x)\right) = i\omega\mathcal{F}\left(f(x)\right)$$

Proof *Using theorem (II.22) we get*

$$\mathcal{F}\left(f'_{gH}(x)\right) = \frac{1}{\sqrt{2\pi}}\left[\left[f(x)e^{i\omega x}\right]_{-\infty}^{\infty} -_g (-i\omega)\int_{-\infty}^{\infty} f(x)e^{i\omega x}dx\right]$$

Using the limite $\lim_{|x|\to\infty} f(x) = 0$ we get the result.

Corollary V.8 *If $f_{gH}^{(k)} \in A^{E^1}$ and $\lim_{|x|\to\infty} f^{(k)(x)} = 0$ for $k = 0, 1, 2$, then*

$$\mathcal{F}\left(f''_{gH}(x)\right) = -\omega^2\mathcal{F}\left(f(x)\right)$$

By the theorems (II.20) and (IV.5) we have

Theorem V.9

$$\mathcal{F}\left({}_{gH}D_t^{\gamma} f(x,t)\right) = {}_{gH}D_t^{\gamma}\mathcal{F}\left(f(x,t)\right)$$

VI. The solution of the fuzzy fractional wave equation

In this section consider the following problem

$$\begin{cases} {}_{gH}D_t^{\gamma}u(t,x) -_g c^2\frac{\partial^2}{\partial x^2}u(t,x) = 0 \\ 0 < x,t < 1,\ 0 < \gamma < 1 \\ u(0,x) = a(x), \\ \frac{\partial}{\partial t}u(0,x) = b(x) \end{cases} \tag{VI.1}$$

where a and b are belongs to A^{E^1},

Proposition VI.1 *the problem (VI.1) has a unique solution.*

Proof *Let $u(x,t)$ is fuzzy absolutely integrable, we define the fuzzy Fourier transform of $u(x,t)$ and its inverse by*

$$\mathcal{F}\left(u(x,t)\right) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} u(x,t)e^{-i\omega t}dx = U(\omega,t)$$

$$\mathcal{F}^{-1}\left(U(\omega,t)\right) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} U(\omega,t)e^{i\omega t}d\omega = u(x,t)$$

If $D_{t_{gH}}^{\gamma}u(x,t)$, $\partial_{x_{gH}}u(x,t)$ and $\partial_{xx_{gH}}u(x,t)$ are fuzzy absolutely integrable in $(-\infty,\infty)$ by using

$$\mathcal{F}\left({}_{gH}D_t^{\gamma}u(t,x)\right) -_g \mathcal{F}\left(c^2\frac{\partial^2}{\partial x^2}u(t,x)\right) = 0$$

It follows from the corollary (V.8) that

$$\mathcal{F}\left(c^2\frac{\partial^2}{\partial x^2}u(t,x)\right) = -c^2\omega^2U(\omega,t)$$

$$\mathcal{F}\left({}_{gH}D_t^{\gamma}u(t,x)\right) = D_t^{\gamma}U(\omega,t)$$

We get

$${}_{gH}D_t^{\gamma}U(\omega,t) = -c^2U(\omega,t)$$

It follows that

$${}_{gH}D_t^{\gamma-1}U'_{gH}(\omega,t) = -c^2U(\omega,t)$$

Thus we have the following problem

$${}_{gH}D_t^{\gamma-1}U'_{gH}(\omega,t) = -c^2U(\omega,t) \tag{VI.2}$$

$$U(\omega,0) = \mathcal{F}\left(a(x)\right) \tag{VI.3}$$

$$\frac{\partial}{\partial t}U(\omega,0) = \mathcal{F}\left(b(x)\right) \tag{VI.4}$$

by lemma 3.2 [5] this problem has a unique solution given by

$$U(\omega,t) = U(\omega,0) + t\frac{\partial}{\partial t}U(\omega,0) -_g \frac{c^2}{\Gamma(\gamma-1)}\int_0^t\int_0^s (s-\tau)^{\gamma-2}U(\omega,\tau)d\tau ds$$

if u' is [(i)-differentiable], and

$$U(\omega,t) = U(\omega,0) + t\frac{\partial}{\partial t}U(\omega,0) + \frac{c^2}{\Gamma(\gamma-1)}\int_0^t\int_0^s (s-\tau)^{\gamma-2}U(\omega,\tau)d\tau ds$$

if u' is [(ii)-differentiable].
Which implies the existence and uniqueness of the solution of the problem (VI.2) and by the inverse of Fourier transform we get the existence and uniqueness of the solution of (VI.1).

VII. Case $\gamma = 2$

In this section we set

$$u(x,t;\alpha) = [\underline{u}(x,t;\alpha), \overline{u}(x,t;\alpha)]$$

$$a(x;\alpha) = [\underline{a}(x;\alpha), \overline{u}(x;\alpha)]$$

$$b(x;\alpha) = [\underline{b}(x;\alpha), \overline{b}(x;\alpha)]$$

If u' is [(i)-differentiable] then

$$\frac{\partial^2}{\partial t^2}\underline{u}(x,t;\alpha) = c^2\frac{\partial^2}{\partial x^2}\underline{u}(x,t;\alpha)$$

$$\frac{\partial^2}{\partial t^2}\overline{u}(x,t;\alpha) = c^2\frac{\partial^2}{\partial x^2}\overline{u}(x,t;\alpha)$$

which implies

$$\underline{u}(x,t;\alpha) = \underline{F}(x-ct;\alpha) + \underline{G}(x+ct;\alpha)$$
$$\overline{u}(x,t;\alpha) = \overline{F}(x-ct;\alpha) + \overline{G}(x+ct;\alpha)$$

where

$$\underline{a}(x;\alpha) = \underline{F}(x-ct;\alpha) + \underline{G}(x+ct;\alpha) \quad \text{(VII.1)}$$

$$\overline{a}(x,t;\alpha) = \overline{F}(x-ct;\alpha) + \overline{G}(x+ct;\alpha) \quad \text{(VII.2)}$$

and

$$\underline{b}(x;\alpha) = \underline{F'}(x-ct;\alpha) + \underline{G'}(x+ct;\alpha) \quad \text{(VII.3)}$$

$$\overline{b}(x,t;\alpha) = \overline{F'}(x-ct;\alpha) + \overline{G'}(x+ct;\alpha) \quad \text{(VII.4)}$$

By the conditions (II.1) the solution is given by

$$u(x,t) = F(x-ct) + G(x+ct)$$

where F and G are given by the above formula $(7.1)-(7.4)$.

VIII. Examples

In this section we will give some examples to illustrate the previous results.

Example VIII.1

$$\begin{cases} {}_{gH}D_t^{\frac{3}{2}}u(t,x) -_g c^2 \frac{\partial^2}{\partial x^2}u(t,x) = 0 \\ 0 < x,t < 1,\ 0 < \gamma < 1 \\ u(0,x;\alpha) = [(1+\alpha)e^{-x^2}, (3-\alpha)e^{-x^2}], \\ \frac{\partial}{\partial t}u(0,x) = 0 \end{cases} \quad \text{(VIII.1)}$$

the solution is given by $u(x,t) = \mathcal{F}^{-1}(U(\omega,t))$ *with*

$$U(\omega,t) = \left[\frac{\alpha+1}{\sqrt{2}}e^{-\omega^2}, \frac{-\alpha+3}{\sqrt{2}}e^{-\omega^2}\right] + \frac{c^2}{\Gamma(\frac{1}{2})}\int_0^t \int_0^s (s-\tau)^{-\frac{1}{4}}U(\omega,\tau)d\tau ds$$

Example VIII.2

$$\begin{cases} {}_{gH}D_t^2u(t,x) -_g c^2 \frac{\partial^2}{\partial x^2}u(t,x) = 0 \\ 0 < x,t < 1,\ 0 < \gamma < 1 \\ u(x,0;\alpha) = [\alpha e^{-x^2}, (2-\alpha)e^{-x^2}], \\ \frac{\partial}{\partial t}u(x,0) = 0 \end{cases} \quad \text{(VIII.2)}$$

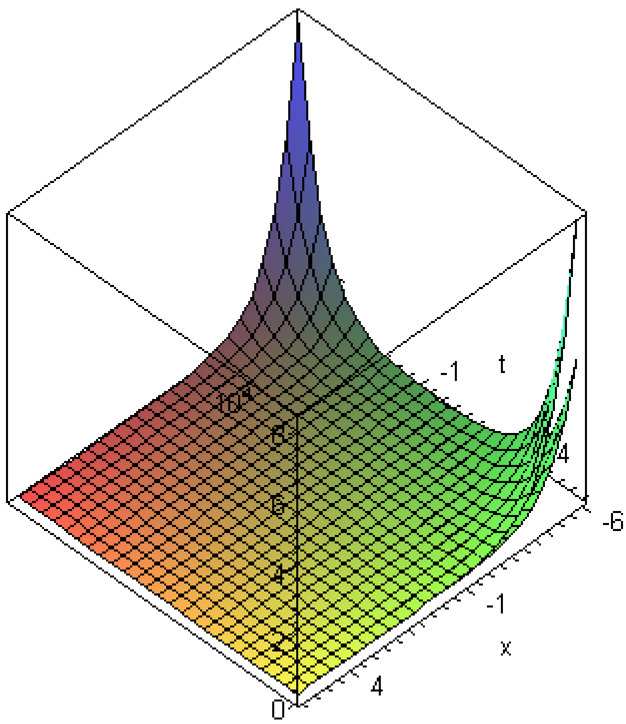

Fig. 1. Lower and upper branch of $u(x,t)$with $\alpha = 1$

the solution is given by

$$u(x,t) = [\alpha, 1-\frac{\alpha}{2}]e^{-x}\cosh(ct)$$

IX. Conclusions

This study makes it possible to explain the wave phenomena with uncertainty in experimental data.

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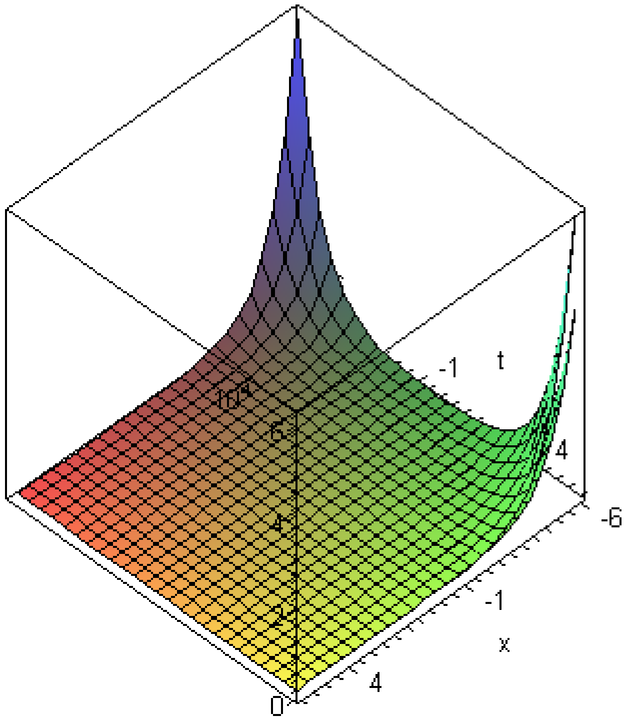

Fig. 2. Lower and upper branch of $u(x,t)$with $\alpha = 0.5$

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