Ulam-Hyers-Rassias stability for fuzzy fractional integrodifferential equations under Caputo gH-differentiability

Said Melliani *, Elhoussain Arhrrabi *, M'hamed Elomari * & Lalla Saadia Chadli *
* Laboratory of Applied Mathematics and Scientific Computing, Sultan Moulay Slimane University, Beni Mellal, Morocco
Email: {said.melliani, arhrrabi.elhoussain, mhamedmaster}@gmail.com
, sa.chadli@hayoo.fr

***Abstract*—In this paper, we establish the Ulam-Hyers stability and Ulam-Hyers-Rassias stability for fuzzy integrodifferential equations under Caputo gH-differentiability by using the fixed point method.**

***Index Terms*—Fuzzy Ulam-Hyers-Rassias stability, Caputo fractional derivatives, fuzzy fractional integrodifferential equations, fixed point theory.**

I. Introduction

In this paper, we will propose fuzzy Ulam-Hyers-Rassias stability for the two kinds of fuzzy fractional integrodifferential equations of order $\alpha \in (0,1)$ with generalized Hukuhara derivative under form

$$\begin{cases} {}_{gH}^{C}\mathcal{D}_{a^+}^{\alpha} u(t) = f(t,u(t)) + \int_a^t g(t,s,u(s))ds, \ t \in [0,a], \\ u(0) = u_0 \in E^d. \end{cases} \tag{1}$$

Where ${}_{gH}^{C}\mathcal{D}_{a^+}^{\alpha}$ is the Caputo's generalized Hukuhara derivative, $f : [0,a] \times E^d \longrightarrow E^d$, is continuous on $[0,a]$ and $g : [0,a] \times [0,a] \times E^d \longrightarrow E$ is continuous on $[0,a] \times [0,a]$. We wish to mention that the theory of fuzzy fractional integral and differential equations have recently been the subject of important studies (see e.g [1]–[11]). In [12], Shen et al studied the Ulam stability problems of the first order linear fuzzy differential equations under some suitable conditions, and in [13], Diaz et al has introduced a fixed point theorem of the alternative for contractions on a generalized metric space, with which Shen et al in [14] proved the Ulam stability of fuzzy differential equations. Since the number of documents dealing with the stability of Ulam for fuzzy fractional integrodifferential equations (FFIEs) is rather limited compared to the number of publications concerning FFIEs, we decide to study by using the fixed point technique, the Ulam-Hyers-Rassias stability for FFIEs.
Our results are inspired by the one in [15] where the fuzzy Ulam-Hyers-Rassias stability of FFIEs is studied. The rest of this paper is organized as follows: In section 2, we recall some notations of the fuzzy number space, the fixed point theorem and the basic notations of the Riemann-Liouville fractional integral and Caputo fractional derivative for fuzzy functions. The Ulam-Hyers-Rassias stability for fuzzy fractional integrodifferential equations are discussed in Sections 3.

II. Preliminaries

In this section, we introduce some definitions, theorems and lemmas which are used in this paper. For more details, we can see papers [3] [9] [12].

Definition 2.1: A function $d : \mathbb{X} \times \mathbb{X} \longrightarrow [0,+\infty)$ is called a generalized metric on $\mathbb{X}$ if and only if d satisfies:

(1) $d(x,y) = 0$ if and only if $x = y$,

(2) $d(x,y) = d(y,x)$ for all $x, y \in \mathbb{X}$,

(3) $d(x,z) \leq d(x,y) + d(y,z)$ for all $x,y,z \in \mathbb{X}$.

Theorem 2.2: $(Banach)$ Let $d : \mathbb{X} \times \mathbb{X} \longrightarrow [0,+\infty)$ be a generalized metric on $\mathbb{X}$ and $(\mathbb{X}, d)$ is a generalized complete metric space. Assume that $T : \mathbb{X} \longrightarrow \mathbb{X}$ is a strictly contractive operator with the Lipschitz constant $L < 1$. If there exists a nonnegative integer n such that $d(T^{n+1}x, T^n x) < \infty$ for some $x \in \mathbb{X}$, then the following are true:

(i) the sequence $T^n x$ converges to a fixed point x^* of T,

(ii) x^* is the unique fixed point of T in $\mathbb{X}^* = \{y \in \mathbb{X} \mid d(T^n x, y) < \infty\}$,

(iii) if $y \in \mathbb{X}^*$, then we have $d(y, x^*) \leq \frac{1}{1-L} d(Ty, y)$.

Lemma 2.3: Let $\varphi : J \longrightarrow [0,+\infty)$ be a continuous function. We define the set

$$\mathbb{X} := \{x : J \longrightarrow \mathbb{R}_{\mathcal{F}} \mid x \ is \ continuous \ function \ on \ J\},$$

where $\mathbb{R}_{\mathcal{F}}$ is the space of fuzzy sets, equipped with the metric $d(x,y) = \inf\{\eta \in [0,+\infty) \cup \{+\infty\} \mid D(x(t),y(t)) \leq \eta\varphi(t), \ \forall t \in J\}$.
Then, $(\mathbb{X}, d)$ is a complete generalized metric space.

Let $K_c(\mathbb{R}^d)$ denote the family of all nonempty, compact and convex subsets of $\mathbb{R}^d$. The addition and scalar multilplication in $K_c(\mathbb{R}^d)$ are defined as usual i.e, for $A, B \in K_c(\mathbb{R}^d)$ and $\lambda\mathbb{R}$,

$$A + B = \{a + b \mid a \in A, b \in B\}, \qquad \lambda A = \{\lambda a \mid a \in A\}$$

Let E^d denote the set of fuzzy subsets of the real axis, if $\omega : \mathbb{R}^d \longrightarrow [0,1]$, satisfying the following properties:
(i) ω is normal, that is, there exists $z_0 \in \mathbb{R}^d$ such that $\omega(z_0) = 1$,

(ii) ω is fuzzy convex, that is, for $0 \leq \lambda \leq 1$

$$\omega(\lambda z_1 + (1-\lambda) z_2) \geq min\{\omega(z_1), \omega(z_2)\},\ for\ any\ z_1, z_2 \in \mathbb{R}^d,$$

(iii) ω is upper semicontinous on $\mathbb{R}^d$,

(iv) $[\omega]^0 = cl\{z \in \mathbb{R}^d : \omega(z) > 0\}$ is compact, where cl denotes the closure in $(\mathbb{R}^d, |\ .\ |)$.
Then E^d is called the space of fuzzy number. For $r \in (0,1]$, we denote $[\omega]^r = \{z \in \mathbb{R}^d \mid \omega(z) \geq r\}$ and $[\omega]^0 = \{z \in \mathbb{R}^d \mid \omega(z) > 0\}$. From the conditions (i) to (iv), it follows that the $r - level$ set of ω, $[\omega]^r$, is a nonempty compact interval, for all $r \in [0,1]$ and any $\omega \in E$.

The notation $[\omega]^r = [\underline{\omega}(r), \overline{\omega}(r)]$, denotes explicitly the $r - level$ set of ω, for $r \in [0,1]$. We refer to $\underline{\omega}$ and $\overline{\omega}$ as the lower and upper branches of ω, respectively. For $\omega \in E^d$, we define the lengh of the $r - level$ set of ω as $len([\omega]^r) = \overline{\omega}(r) - \underline{\omega}(r)$. For addition and scalar multiplication in fuzzy set space E^d, we have $[\omega_1 + \omega_2]^r = [\omega_1]^r + [\omega_2]^r$, $[\lambda\omega]^r = \lambda[\omega]^r$.

The Hausdorff distance between fuzzy numbers is given by

$$D_0[\omega_1, \omega_2] = \sup_{0 \leq r \leq 1} \{|\ \underline{\omega}_1(r) - \underline{\omega}_2(r)\ |, |\ \overline{\omega}_1(r) - \overline{\omega}_2(r)\ |\}.$$

The metric space (E^d, D_0) is complet metric space and the following properties of the metric D_0 are valid.

$$D_0[\omega_1 + \omega_3, \omega_2 + \omega_3] = D_0[\omega_1, \omega_2],$$

$$D_0[\lambda\omega_1, \lambda\omega_2] = |\ \lambda\ |\ D_0[\omega_1, \omega_2],$$

$$D_0[\omega_1, \omega_2] \leq D_0[\omega_1, \omega_3] + D_0[\omega_3, \omega_2],$$

for all $\omega_1, \omega_2, \omega_3 \in E^d$ and $\lambda \in \mathbb{R}^d$. Let $\omega_1, \omega_2 \in E^d$, if there exists $\omega_3 \in E^d$ such that $\omega_1 = \omega_2 + \omega_3$ then ω_3 is called the H-difference of ω_1, ω_2. We denote the ω_3 by $\omega_1 \ominus \omega_2$. Let us remark that $\omega_1 \ominus \omega_2 \neq \omega_1 + (-1)\omega_2$.

Definition 2.4: The generalized Hukuhara difference of two fuzzy numbers $\omega_1, \omega_2 \in E^d$ (gH-difference for short) is defined as follows:

$$\omega_1 \ominus_{gH} \omega_2 = \omega_3 \Leftrightarrow \begin{cases} (i)\ \omega_1 = \omega_2 + \omega_3, \\ or\ (ii)\ \omega_2 = \omega_1 + (-1)\omega_3. \end{cases}$$

Let $[0,a]$ be a compact interval in $\mathbb{R}^+$. Denote by $diam[u(t)]^r$ the diameter of fuzzy set u, for $t \in [0,a]$. A function $u : [0,a] \longrightarrow E^d$ is called ω-increasing (ω-decreasing) on $[0,a]$ if for every $r \in [0,1]$ the function $t \longmapsto diam[u(t)]^r$ is nondecreasing (nonincreasing) on $[0,a]$. If u is ω-increasing or ω-decreasing on $[0,a]$, then we say that u is ω-monotone on $[0,a]$.

Definition 2.5:

Let $t \in (a,b)$ and h such that $t + h \in (a,b)$, then the generalized Hukuhara derivative of fuzzy-valued function $x : (a,b) \longrightarrow E^d$ at t is defined as

$$D_{gH}x(t) = \lim_{h \longrightarrow 0} \frac{x(t+h) \ominus_{gH} x(t)}{h}.$$

If $D_{gH}x(t) \in E^d$ satisfying last inequality, we say that x is generalized Hukuhara differentiable (gH-differentiable for short) at t.

Definition 2.6: Let $x : [a,b] \longrightarrow E^d$, the fuzzy Rieman-Liouville integral of fuzzy-valued function x is defined as follows:

$$(\mathcal{J}^{\alpha}_{a^+}x)(t) = \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} x(s) ds.$$

For $a \leq t$, and $0 < \alpha \leq 1$. For $=\alpha = 1$, we set $\mathcal{J}^1_a = I$, the identity operator.

Definition 2.7: Let $D_{gH} \in C([a,b], E^d) \cap L([a,b], E^d)$. The fuzzy gH-fractional Caputo diffentiability of fuzzy-valued function x $([gH]^C_a - differentiable\ for\ short)$ is defined as following:
${}^{C}_{gH}\mathcal{D}^{\alpha}_{a^+}x(t) = \mathcal{J}^{1-\alpha}_{a^+}(D_{gH}x)(t) = \frac{1}{\Gamma(1-\alpha)} \int_a^t (t-s)^{-\alpha} (D_{gH}x)(s) ds$,
where $0 < \alpha \leq 1$, $t > a$.

Lemma 2.8: Suppose that $x : [a,b] \longrightarrow E^d$ be a fuzzy function and $D_{gH}x(t) \in C([a,b], E^d) \cap L([a,b], E^d)$. Then

$$\mathcal{J}^{\alpha}_{a^+}({}^{C}_{gH}\mathcal{D}^{\alpha}_{a^+}x)(t) = x(t) \ominus_{gH} x(a).$$

Lemma 2.9: Let $u : [0,a] \longrightarrow E^d$ be a continuous function on $[0,a]$ and let $\alpha \in (0,1)$, then the FFIE (1) is equivalent to the following integral equation:

(1) If u is ω-increasing on $[0,a]$, then

$$u(t) = \varphi(0) + \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} (f(s, u(s)) + \int_a^s g(s, r, u(r)) dr) ds, \quad (2)$$

(2) If u is ω-decreasing on $[0,a]$, then

$$u(t) = \varphi(0) \ominus \frac{(-1)}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} (f(s, u(s)) + \int_a^s g(s, r, u(r)) dr) ds. \quad (3)$$

III. Main results

In the sequel, our aim score is to present the results for the existence and the stability of the problem (1). The methods to solve these problems are quite similar. However, since the conditions for the existence of solutions of fuzzy fractional integrodifferential equations (2) and (3) are dissimilar, we shall present the two kinds (2) and (3) in two separate subsections.

International Journal on Optimization and Applications
IJOA. Vol. 1, Issue No. 2, Year 2021, www.usms.ac.ma/ijoa

A. Fuzzy Ulam-Hyers-Rassias stability for FFIEs (2)

Firstly, we present the definitions of fuzzy Ulam-Hyers stability and fuzzy Ulam-Hyers-Rassias stability.

Definition 3.1: We say that the problem (2) is fuzzy Ulam-Hyers stable, if there exists a constant $K_f > 0$ such that for each $\varepsilon > 0$ and for each solution $v \in C^1([0,a], E^d)$ of the following inequality
$D\left[{}^{C}_{gH}\mathcal{D}^{\alpha}_{a^+}v(t), f(t,v(t)) + \int_a^t g(t,s,v(s))ds\right] \leq \varepsilon, \forall t \in [0,a]$,
then, there exists a solution $u \in C^1([0,a], E^d)$ of problem (2) with

$$D\left[v(t), u(t)\right] \leq K_f\varepsilon,$$

for all $t \in [0,a]$. We call K_f a Ulam-Hyers stability constant of (2).

Definition 3.2: We say that the problem (2) is fuzzy Ulam-Hyers-Rassias stable, if there exists a constant $C_f > 0$ such that for each $\varepsilon > 0$ and for each solution $v \in C^1([0,a], E^d)$ of the following inequality
$D\left[{}^{C}_{gH}\mathcal{D}^{\alpha}_{a^+}v(t), f(t,v(t)) + \int_a^t g(t,s,v(s))ds\right] \leq \varphi(t), \forall t \in [0,a]$,
then, there exists a solution $u \in C^1([0,a], E^d)$ of problem (2) with

$$D\left[v(t), u(t)\right] \leq C_f\varphi(t),$$

for all $t \in [0,a]$ and for some nonnegative function φ defined on $[0,a]$.

Remark 3.3: We observe that definition 3.2 $\Rightarrow$ definition 3.1.

In the following, we shall prove that the FFIEs (2) is fuzzy Ulam-Hyers-Rassias stable on bounded interval by the fixed point theorem.

Theorem 3.4: Assume that $f : [0,a] \times E^d \longrightarrow E^d$ and $g : [0,a] \times [0,a] \times E^d \longrightarrow E^d$ are continuous functions satisfying the following conditions:

(i) There exists a constant $L_{fg} > 0$ such that:

$$\max\left\{D[f(t,u), f(t,v)]; D[g(t,s,u), g(t,s,v)]\right\} \leq L_{fg}D[u,v], \tag{4}$$

for all each $(t,s,u), (t,s,v) \in [0,a] \times [0,a] \times E^d$.

(ii) There exists a constant $K, C > 0$ such that $0 < L_{fg}K(1+C) < 1$ and let $\varphi : [0,a] \longrightarrow [0,\infty)$ be a continuous function and increasing on $[0,a]$ with:

$$\int_a^t \varphi(s)ds \leq C.\varphi(t), \qquad \forall t \in [0,a], \tag{5}$$

and

$$\frac{1}{\Gamma(\alpha)}\int_a^t (t-s)^{\alpha-1}\varphi(s)ds \leq K\varphi(t), \qquad \forall t \in [0,a], \tag{6}$$

If a continuously ω-increasing function $u : [0,a] \longrightarrow E^d$ satisfies the following inequality

$$D\left[{}^{C}_{gH}\mathcal{D}^{\alpha}_{a^+}u(t), f(t,u(t)) + \int_a^t g(t,s,u(s))ds\right] \leq \varphi(t), \tag{7}$$

for any $t \in [0,a]$, then there exists a unique $\tilde{u} : [0,a] \longrightarrow E^d$ of (2.2) such that

$$\tilde{u}(t) = u_0 + \frac{1}{\Gamma(\alpha)}\int_a^t (t-s)^{\alpha-1}(f(s,\tilde{u}(s)) + \int_a^s g(s,r,\tilde{u}(r))dr)ds, \tag{8}$$

and

$$d[\tilde{u}(t), u(t)] \leq \frac{1}{1 - L_{fg}K(1+C)}, \qquad \forall t \in [0,a]. \tag{9}$$

Proof:

Let us consider the space of all continuous fuzzy function $u : [0,a] \longrightarrow E^d$ by

$$\mathbb{X} = \{u : [0,a] \longrightarrow E^d \mid u \ is\ continuous\ on\ [0,a]\},$$

equipped by the metric

$d(u,v) = \inf\{C \in [0,+\infty) \cup \{+\infty\} \mid D[u(t),v(t)] \leq C\varphi(t)\},\ \forall t \in [0,a]$.

By lemma 2.3, we observe that $(\mathbb{X}, d)$ is also a complete generalized metric space. We define an operator $Q : \mathbb{X} \longrightarrow \mathbb{X}$ by

$$(Qu)(t) = u_0 + \frac{1}{\Gamma(\alpha)}\int_a^t (t-s)^{\alpha-1}(f(s,u(s)) + \int_a^s g(s,r,u(r))dr)ds, \qquad \forall t \in [0,a]. \tag{10}$$

Because f and g are a continuous fuzzy functions, the right hand side of (10) is also continuous on $[0,a]$. This yields that Qu is continuous on $[0,a]$. So, the operator Q is well-defined. To apply theorem 2.2 in the proof of this theorem, we need the operator Q to be strict contractive on $\mathbb{X}$. For any $u, v \in \mathbb{X}$ and let $C_{uv} \in [0,+\infty) \cup \{+\infty\}$ such that

$$d(u,v) \leq C_{uv}, \qquad \forall t \in [0,a].$$

Then, by the definition of d, we have

$$D[u(t), v(t)] \leq C_{uv}\varphi(t), \qquad \forall t \in [0,a]. \tag{11}$$

From the definition of the operator Q and assumption (4)-

(6), we have the following estimation

$$
\begin{aligned}
&D[(Qu)(t),(Qv)(t)] = D[u_0 + \frac{1}{\Gamma(\alpha)}\int_a^t (t-s)^{\alpha-1}(f(s,u(s))\\
&+\int_a^s g(s,r,u(r))dr)ds, u_0 + \frac{1}{\Gamma(\alpha)}\int_a^t (t-s)^{\alpha-1}(f(s,v(s))\\
&+\int_a^s g(s,r,v(r))dr)ds],\\
&\leq \frac{1}{\Gamma(\alpha)}\int_a^t (t-s)^{\alpha-1} D[f(s,u(s)),f(s,v(s))]ds\\
&+\frac{1}{\Gamma(\alpha)}\int_a^t (t-s)^{\alpha-1}(\int_a^s D[g(s,r,u(r)),g(s,r,v(r))]dr)ds,\\
&\leq \frac{L_{fg}}{\Gamma(\alpha)}\int_a^t (t-s)^{\alpha-1} D[u(s),v(s)]ds\\
&+\frac{L_{fg}}{\Gamma(\alpha)}\int_a^t (t-s)^{\alpha-1}(\int_a^s D[u(r),v(r)]dr)ds,\\
&\leq \frac{L_{fg}C_{uv}}{\Gamma(\alpha)}\int_a^t (t-s)^{\alpha-1}\varphi(s)ds\\
&+\frac{L_{fg}C_{uv}}{\Gamma(\alpha)}\int_a^t (t-s)^{\alpha-1}(\int_a^s \varphi(r)dr)ds,\\
&\leq L_{fg}C_{uv}K\varphi(t) + L_{fg}C_{uv}CK\varphi(t)\\
&= L_{fg}K(1+C)C_{uv}\varphi(t).
\end{aligned}
$$

Hence

$$D[(Qu)(t),(Qv)(t)] \leq L_{fg}K(1+C)C_{uv}\varphi(t). \tag{12}$$

So, by the definition of metric d, we get

$$d(Qu,Qv) \leq L_{fg}K(1+C)d(u,v), \quad for\ all\ u,v \in E^d.$$

Where $0 < L_{fg}K(1+C) < 1$, hence the operator Q is strictly contractive mapping on $\mathbb{X}$.
For an arbitrary $\omega \in \mathbb{X}$ and from the definition of $\mathbb{X}$ and Q, it follows that there exists a constant $0 < C_\omega < \infty$ such that:

$$
\begin{aligned}
D[(Q\omega)(t),\omega(t)] = D[u_0 + \tfrac{1}{\Gamma(\alpha)}\textstyle\int_a^t (t-s)^{\alpha-1}(f(s,\omega(s))\\
+\textstyle\int_a^s g(s,r,\omega(r))dr)ds, \omega(t)] \leq C_\omega\varphi(t),
\end{aligned}
$$

for any $t \in [0,a]$, since f, g and ω are bounded on $[0,a]$, and the minimum of $\varphi(t) > 0$ on $t \in [0,a]$. Then, we infer that $d(Q\omega,\omega) \leq C_\omega < \infty$. Therefore, according to (i) and (ii) of theorem 2.2, there exists a continuously function $\tilde{u} : [0,a] \longrightarrow E^d$ such that $Q^n\omega \longrightarrow \tilde{u}$ in the space $(\mathbb{X}, d)$ as $n \longrightarrow \infty$ and $Q\tilde{u} = \tilde{u}$, that $\tilde{u}$ satisfies the problem (8) for any $t \in [0,a]$.
Now, we shall confirm that $\{u \in \mathbb{X} \mid d(\omega,u) < \infty\} = \mathbb{X}^*$. For an arbitrary $u \in E^d$, since u and ω are bounded on $[0,a]$ and $\min_{t\in[0,a]} \varphi(t) > 0$, there exists a constant $0 < C_u < \infty$ such that $D[\omega(t),u(t)] \leq C_u\varphi(t)$ for any $t \in [0,a]$. Therefore, we have $d(\omega,u) < \infty$ for any $u \in E^d$, that is $\{u \in \mathbb{X} \mid d(\omega,u) < \infty\} = \mathbb{X}^*$. By theorem 2.2-$(ii)$, we conclude that $\tilde{u}$ is the unique fixed point of Q on $\mathbb{X}$.
On the other hand, from the inequality (7) it follows that

$$d(u,Qu) \leq 1. \tag{13}$$

Finally, by theorem 2.2 $-$ (iii) and from the estimation (13), it implies that

$$d(\tilde{u}(t),u(t)) \leq \frac{d(u,Qu)}{1 - L_{fg}K(1+C)} \leq \frac{1}{1 - L_{fg}K(1+C)},$$

which means the estimation (9) holds true for any $t \in [0,a]$. This completes the proof. □

B. Fuzzy Ulam-Hyers-Rassias stability for FFIEs (3)

Theorem 3.5: Suppose that the functions f, g and φ satisfy all conditions as in theorem 3.4. Assume that for each $t \in [0,a]$ and for each continuous fuzzy function $z : [0,a] \longrightarrow E^d$, if the Hukuhara difference $z(0) \ominus \frac{(-1)}{\Gamma(\alpha)}\int_a^t (t-s)^{\alpha-1}\left(f(s,z(s)) + \int_a^s g(s,r,z(r))dr\right)ds$, exists and a continuously ω-nonincreasing function $v : [0,a] \longrightarrow E^d$ satisfies

$$
\begin{aligned}
D[v(t), v_0 \ominus \frac{(-1)}{\Gamma(\alpha)}\int_a^t (t-s)^{\alpha-1}(f(s,v(s))\\
+\int_a^s g(s,r,v(r))dr)ds] \leq \varphi(t),
\end{aligned} \tag{14}
$$

for any $t \in [0,a]$, where $v_0 = u_0$, then there exists a unique solution $\hat{u} : [0,a] \longrightarrow E^d$ of the problem (3) which satisfies

$$
\begin{aligned}
\hat{u}(t) = u_0 \ominus \frac{(-1)}{\Gamma(\alpha)}\int_a^t (t-s)^{\alpha-1}(f(s,\hat{u}(s))\\
+\int_a^s g(s,r,\hat{u}(r))dr)ds,
\end{aligned} \tag{15}
$$

and

$$d[\hat{u}(t),v(t)] \leq \frac{1}{1 - L_{fg}K(1+C)}, \tag{16}$$

for any $t \in [0,a]$.

Proof:
We consider the complete generalized space $(\mathbb{X}, d)$ defined as in the proof of theorem 2. Define the operator $P : \mathbb{X} \longrightarrow \mathbb{X}$ as follows:

$$
\begin{aligned}
(Pu)(t) = u_0 \ominus \frac{(-1)}{\Gamma(\alpha)}\int_a^t (t-s)^{\alpha-1}(f(s,u(s))\\
+\int_a^s g(s,r,u(r))dr)ds, \quad t \in [0,a].
\end{aligned} \tag{17}
$$

Since the function f and g is continuous on $[0,a]$ and the Hukuhara difference $u_0 \ominus \frac{(-1)}{\Gamma(\alpha)}\int_a^t (t-s)^{\alpha-1}\left(f(s,u(s)) + \int_a^s g(s,r,u(r))dr\right)ds$ exists, similary to theorem 1, it follows that Pu is well-defined on $[0,a]$ or Pu is continuous on $[0,a]$. Now, we observe that the operator P is strictly contractive on $\mathbb{X}$. Indeed, for any $u, v \in \mathbb{X}$ and let $C_{uv} \in [0,+\infty) \cup \{+\infty\}$ be an arbitrary constant with $d(u,v) \leq C_{uv}$ for $t \in [0,a]$, that is, let us assume that

$$D[u(t),v(t)] \leq C_{uv}\varphi(t), \tag{18}$$

for $t \in [0, a]$. Furthermore, from (17), (18) and by the Lipschitz condition of f and g, we have the following estimation:

$$D[(Pu)(t), (Pv)(t)] = D[u_0 \ominus \frac{(-1)}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} (f(s, u(s)) + \int_a^s g(s, r, u(r)) dr) ds, u_0 \ominus \frac{(-1)}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} (f(s, v(s)) + \int_a^s g(s, r, v(r)) dr) ds],$$
$$\leq \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} D[f(s, u(s)), f(s, v(s))] ds + \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} (\int_a^s D[g(s, r, u(r)), g(s, r, v(r))] dr) ds,$$
$$\leq \frac{L_{fg}}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} D[u(s), v(s)] ds + \frac{L_{fg}}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} (\int_a^s D[u(r), v(r)] dr) ds,$$
$$\leq L_{fg} C_{uv} K \varphi(t) + L_{fg} C_{uv} C K \varphi(t)$$
$$= L_{fg} K (1 + C) C_{uv} \varphi(t).$$

Hence

$$D[(Pu)(t), (Pv)(t)] \leq L_{fg} K (1 + C) C_{uv} \varphi(t). \tag{19}$$

This means that $d(Pu, Pv) \leq L_{fg} K (1 + C) d(u, v)$. Hence, the operator P is a strictly contractive mapping on $\mathbb{X}$ by the assumption $0 < L_{fg} K (1 + C) < 1$. Simalar to the theorem 3.4, we can show that for each $\omega \in \mathbb{X}$ satisfies $d(P\omega, \omega) < \infty$. Hence, by theorem 1, it implies that there exists a continuously function $\hat{u} : [0, a] \longrightarrow E^d$ such that $P^n \omega \longrightarrow \hat{u}$ in $(\mathbb{X}, d)$ as $n \longrightarrow \infty$, and such that $P\hat{u} = \hat{u}$, that is $\hat{u}$ satisfies (4.15) for $t \in [0, a]$. Similar to the proof of theorem 3.4, we observe that there exists a constant $C_\omega > 0$ such that $D[\omega(t), u(t)] \leq C_\omega$, for any $t \in [0, a]$. This means that $d(\omega, u) < \infty$ for each $u \in E^d$, or equivalently, $\{u \in \mathbb{X} \mid d(\omega, u) < \infty\} = \mathbb{X}^*$. Furthermore, by theorem 2.2, we imply that $\hat{u}$ is a unique continuous function which satisfies (15).
Moreover, by theorem 2.2, we also obtain

$$d(\hat{u}(t), u(t)) \leq \frac{d(u, Pu)}{1 - L_{fg} K (1 + C)} \leq \frac{1}{1 - L_{fg} K (1 + C)},$$

which means the estimation (16) holds true for any $t \in [0, a]$. This completes the proof. □

IV. Conclusion

In this study, we are studied the Ulam-Hyers-Rassias stability for fuzzy intergodifferential equation via the fixed point technique. This result can be used to study fractional fuzzy differential equations with other types of derivative concepts in fuzzy setting, for example, Riemann-Liouville and Hadamard generalized Hukuhara differentiability.

References

[1] T. V. An, H. Vu, N. V. Hoa, A new technique to solve the initial value problems for fractional fuzzy delay differential equations, Advances in Difference Equations, 181 (2017),

[2] T. V. An, H. Vu, N. V. Hoa, Applications of contractive-like mapping principles to interval-valued fractional integro-differential equations, Journal of Fixed Point Theory and Applications, 19(4) (2017), 2577-2599.

[3] T. V. An, H. Vu, N. V. Hoa, Hadamard-type fractional calculus for fuzzy functions and existence theory for fuzzy fractional functional integro-differential equations, Journal of Intelligent and Fuzzy Systems, 36(4) (2019), 3591-3605.

[4] N. V. Hoa, Fuzzy fractional functional differential equations under Caputo gH-differentiability, Communications in Nonlinear Science and Numerical Simulation, 22(1) (2015), 1134-1157.

[5] N. V. Hoa, Fuzzy fractional functional integral and differential equations, Fuzzy Sets and Systems, 280 (2015), 58-90.

[6] N. V. Hoa, V. Lupulescu, D. Oregan, Solving interval-valued fractional initial value problems under Caputo gH-fractional differentiability, Fuzzy Sets and Systems, 309 (2017), 1-34.

[7] N. V. Hoa, H. Vu, T. Minh Duc, Fuzzy fractional differential equations under Caputo-Katugampola fractional derivative approach, Fuzzy Sets and Systems, 375(15) (2019), 70-99.

[8] H. V. Long, N. T. K. Son, N. V. Hoa, Fuzzy fractional partial differential equationd in partially ordered metric spaces, Iranian Journal of Fuzzy Systems, 14(2) (2017), 107-126.

[9] M. T. Malinowski, Random fuzzy fractional integral equations-theoretical foundations, Fuzzy Sets and Systems, 265 (2015), 39-62.

[10] N. Ngoc Phung, B. Quoc Ta, H. Vu, Hyers-Ulam stability and Hyers-Ulam-Rassias stability for fuzzy integrodifferential equation, Complexity, 2019 (2019),

[11] H. Vu, V. Lupulescu, N. Van Hoa, Existence of extremal solutions to interval-valued delay fractional differential equations via monotone iterative technique, Journal of Intelligent and Fuzzy Systems, 34(4) (2018), 2177-2195.

[12] Y. Shen, On the Ulam stability of
rst order linear fuzzy differential equations under generalized differentiability, Fuzzy Sets and Systems, 280(C) (2015), 27-57.

[13] J. B. Diaz, B. Margolis, A
fixed point theorem of the alternative, for contractions on a generalized complete metric space, Bulletin of the American Mathematical Society, 74(2) (1968), 305-309.

[14] Y. Shen, On the Ulam stability of first order linear fuzzy differential equations under generalized differentiability, Fuzzy Sets and Systems, 280(C) (2015), 27-57.

[15] H. Vu, J. M. Rassias and N. Van Hoa, Ulam-Hyers-Rassias stability for fuzzy fractional integral equations, Iranian Journal of Fuzzy Systems Volume 17, Number 2, (2020), pp. 17-27.