On the existence result of fuzzy fractional boundary value problems

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***Abstract*—In this paper we investigate the existence result of solutions for boundary value problem of nonlinear fuzzy fractional differential equations involving Caputo fuzzy fractional derivatives. We conclude our work by presenting an illustrative example.**

***Index Terms*—Fuzzy numbers,Fuzzy functions,fuzzy fractional integral,fuzzy fractional caputo derivative.**

I. Introduction

Fractional differential equations (DEs) have received considerable attention in the recent years due to their wide applications in the areas of applied mathematics, physics, engineering, economy, and other fields. Many important phenomena in electromagnetic, acoustics, viscoelasticity, electrochemistry, and material science are well described by fractional DE [6], [7], [8], [9], [10], [11].In general, most of fractional DEs do not have exact solutions. Particularly, there is no known method for solving fractional boundary value problems (BVPs) exactly. As a result, numerical and analytical techniques have been used to study such problems. It should be noted that much of the work published to date concerning exact and numerical solutions is devoted to the initial value problems for fractional order ordinary DEs. The theory of BVPs for fractional DEs has received attention quiet recently. The attention drawn to the theory of existence and uniqueness of solutions to BVPs for fractional order DEs is evident from the increased number of recent publications.In the book by Kelley and Peterson [1] the following result is established:

Theorem 1: ([1],Theorem 7.7).Assume $f : [a; b] \times R \longrightarrow R$ is continuous and satisfies a uniform Lipschitz condition with respect to the second variable on $[a; b] \times R$ with Lipschitz constant K; that is,

$$| f(t, x) - f(t, y) | \leq K | x - y |$$

for all $(t, x), (t, y) \in [a; b] \times R$.if

$$b - a < \frac{2\sqrt{2}}{\sqrt{K}}$$

then the boundary valued problem

$$y''(t) = f(t, y(t)), \quad t \in [a, b]$$

$$y(a) = A, y(b) = B \quad A, B \in R$$

has a unique continuous solution.

In this work we want to extend the above result by considering a fractional Riemmann-Liouville derivative (we refer the reader to [5] for the defnitions and basic results on fractional calculus) instead of the classical operator y'',i.e., we prove the existence and uniqueness of solutions for the fuzzy fractional diferential boundary value problem.

$$D^{\alpha} x(t) = f(t, x(t)) \quad t \in [a, b] \tag{I.1}$$

$$x(a) = \tilde{0}, x(b) = B \quad B \in E^1 \tag{I.2}$$

Where $1 < \alpha \leq 2$ and E^1 is the collection of fuzzy numbers.

II. Preliminaries

Definition 1: [4] A fuzzy number is mapping $u : \mathbb{R} \longrightarrow [0, 1]$ such that

1) u is upper semi-continuous
2) u is normal,that is, there exist $x_0 \in \mathbb{R}$ such that $u(x_0) = 1$
3) u is fuzzy convex, that is, $u(\lambda x + (1 - \lambda)y) \geq \underline{\min\{u(x), u(y)\}}$ for all $x, y \in \mathbb{R}$ and $\lambda \in [0, 1]$.
4) $\overline{\{x \in \mathbb{R},\ u(x) > 0\}}$ is compact.

The $\alpha - Cut$ of a fuzzy number u is defined as follows:

$$[u]^{\alpha} = \{x \in \mathbb{R}/u(x) \geq \alpha\}$$

We denote by E^1 the collection of all fuzzy numbers.

Definition 2: [4] A fuzzy number u in a parametric form is a pair of function $(\underline{u}(r), \overline{u}(r))$ with $r \in [0, 1]$,which satisfy the following requirements:

1) $\underline{u}(r)$ is a bounded nonincreasing lef continuous function in $[0, 1]$.
2) $\overline{u}(r)$ is a bounded nondecreasing lef continuous function in $[0, 1]$.
3) $\underline{u}(r) \leq \overline{u}(r)\ \forall r \in [0, 1]$

Moreover,we also can present the $r - cut$ representation of fuzzy number as $[u]^r = [\underline{u}(r), \overline{u}(r)]$.

Definition 3: Let $x, y \in E^1$,if there exists $z \in E^1$ such that,$x = y + z$ then z is called the Hukuhara difference of x and y, denoted by $x \ominus y$.

Definition 4: According to the Zadeh's extension principle, the addition on E^1 is defined by

$$(u \oplus v)(z) := \sup_{z=x+y} min\{u(x), v(y)\}$$

And scalar multiplication of a fuzzy number is given by

$$(k \odot u)(x) := \begin{cases} u(x/k) & , k > 0 \\ \widetilde{0} & , k = 0 \end{cases}$$

Definition 5: [4] Let $f : [a, b] \to E^1$ and $t_0 \in [a, b]$.We say that f is Hukuhara differentiable at t_0 if there exists $f'(t_0) \in E^1$ such that:
(1) $f'(t_0) = \lim_{h\to 0^+} \frac{f(t_0+h)\ominus f(t_0)}{h}$
$= \lim_{h\to 0^-} \frac{f(t_0)\ominus f(t_0-h)}{h}$
Or
(2) $f'(t_0) = \lim_{h\to 0^+} \frac{f(t_0)\ominus f(t_0+h)}{-h}$
$= \lim_{h\to 0^-} \frac{f(t_0-h)\ominus f(t_0)}{-h}$

Proposition 1: Let $f : [a, b] \longrightarrow E^1$ be a function such that $[f(x)]^r = \left[\underline{f}(x;r), \overline{f}(x;r)\right]$ for each $r \in [0, 1]$

1) If f is (1)-differentiable function,then $[f'(x)]^r = \left[\underline{f}'(x;r), \overline{f}'(x;r)\right]$
2) If f is (2)-differentiable function,then $[f'(x)]^r = \left[\overline{f}'(x;r), \underline{f}'(x;r)\right]$

Definition 6: [4] Let $u = (\underline{u}(r), \overline{u}(r)), v = (\underline{v}(r), \overline{v}(r)) \in E^1$ with $r \in [0, 1]$,then the Hausdorf distance between u and $v$is given by
$D(u, v) = \sup_{r\in[0,1]} \max\{|\underline{u}(r) - \underline{v}(r)|, |\overline{u}(r) - \overline{v}(r)|\}$

Proposition 2: [4] D is a metric on E^1 and has the following properties:

1) $(E^1; D)$ is a complete metric space.
2) $D(u + w, v + w) = D(u, v)$, $\forall u, v, w \in E^1$.
3) $D(ku, kv) = |k|D(u, v)$, $\forall u, v \in E^1$ and $k \in \mathbb{R}$.
4) $D(u+w, v+z) \leq D(u, v)+D(w, z)$, $\forall u, v, w, z \in E^1$.

We denote by $\mathcal{C}^F = \mathcal{C}([a, b], E^1)$ space of all fuzzy-valued functions which are continuous on $[a, b]$,and $\mathcal{P}_K(\mathbb{R})$ is the collection of all the compact subset of $\mathbb{R}$.

Definition 7: $F : [a, b] \longrightarrow E^1$ is strongly measurable if $\forall\alpha \in [0, 1]$,the set-valued mapping $F_\alpha : [a, b] \longrightarrow \mathcal{P}_K(\mathbb{R})$ defined by $F_\alpha(t) = [F(t)]^\alpha$ is Lebesgue measurable A function $F : [a, b] \longrightarrow E^1$ is called integrably bounded, if there exists an integrable function h such that $|x| < h(t)$ $\forall x \in F_0(t)$.

Definition 8: Let $F : [a, b] \longrightarrow E^1$. The integral of F on $[a, b]$ denoted by $\displaystyle\int_a^b F(t)dt$,is given by
$\left[\int_a^b F(t)dt\right]^\alpha =$
$\left\{\int_a^b f(t)dt \mid f : [a, b] \longrightarrow \mathbb{R}\text{is a measurable selection for}F_\alpha\right\}$
for all $\alpha \in [0, 1]$.

Definition 9: [5] Let $f : [a, b] \longrightarrow E^1$ and $0 < \alpha < 1$,the fuzzy Riemann-Liouville fractional integral is defined by

$$I^\alpha_{a+} f(t) := \frac{1}{\Gamma(\alpha)} \int_a^t \frac{f(s)}{(t - s)^{1-\alpha}} ds$$

Remark 1: Since$[f(x)]^r = \left[\underline{f}(x;r), \overline{f}(x;r)\right]$ for each $r \in [0, 1]$,then
$\left[I^\alpha_{a+} f(t)\right]^r = \left[I^\alpha_{a+}\underline{f}(x;r), I^\alpha_{a+}\overline{f}(x;r)\right]$
Where $I^\alpha_{a+}\underline{f}(t;r) := \frac{1}{\Gamma(\alpha)} \int_a^t \frac{\underline{f}(s;r)}{(t-s)^{1-\alpha}} ds$
And
$I^\alpha_{a+}\overline{f}(t;r) := \frac{1}{\Gamma(\alpha)} \int_a^t \frac{\overline{f}(s;r)}{(t-s)^{1-\alpha}} ds$

Definition 10: [5] Let $f : [a, b] \longrightarrow E^1$,$x_0 \in [a, b]$ and $\phi(x) = \dfrac{1}{\Gamma(\alpha)} \displaystyle\int_a^x \frac{f(s)}{(t - s)^{1-\alpha}} ds.$
The function f is called fuzzy Riemann-Liouville fractional differentiabl of order $0 < \alpha < 1$ at x_0 if there exists an element $D^\alpha_a f(x_0) \in E^1$ such that
(1) $D^\alpha_a f(x_0) = \lim_{h\to 0^+} \frac{\phi(x_0+h)\ominus\phi(x_0)}{h}$
$= \lim_{h\to 0^-} \frac{\phi(x_0)\ominus\phi(x_0-h)}{h}$
Or
(2) $D^\alpha_a f(x_0) = \lim_{h\to 0^+} \frac{\phi(x_0)\ominus\phi(x_0+h)}{-h}$
$= \lim_{h\to 0^-} \frac{\phi(x_0-h)\ominus\phi(x_0)}{-h}$
For the sake of simplicity, a fuzzy-valued function f is ${}^{RL}[(1) - \alpha] - differentiable$ if it is differentiable,as in definition(2.6), Case (1), and is ${}^{RL}[(2) - \alpha] - differentiable$ if it is differentiable as in definition (2.6),Case (2).

Remark 2: [5] Since$[f(x)]^r = \left[\underline{f}(x;r), \overline{f}(x;r)\right]$ for each $r \in [0, 1]$,then we have the following relations:
1)If f is ${}^{RL}[(1) - \alpha] - differentiable$fuzzy valued function then,
$\left[D^\alpha_{a+} f(t)\right]^r = \left[D^\alpha_{a+}\underline{f}(x;r), D^\alpha_{a+}\overline{f}(x;r)\right]$
1)If f is ${}^{RL}[(2) - \alpha] - differentiable$fuzzy valued function then,
$\left[D^\alpha_{a+} f(t)\right]^r = \left[D^\alpha_{a+}\overline{f}(x;r), D^\alpha_{a+}\underline{f}(x;r)\right]$
Where $D^\alpha_{a+}\underline{f}(t;r) := \frac{1}{\Gamma(\alpha)} \frac{d}{dt} \int_a^t \frac{\underline{f}(s;r)}{(t-s)^{1-\alpha}} ds$
And
$D^\alpha_{a+}\overline{f}(t;r) := \frac{1}{\Gamma(\alpha)} \frac{d}{dt} \int_a^t \frac{\overline{f}(s;r)}{(t-s)^{1-\alpha}} ds$

Definition 11: [5] Let $f : [a, b] \longrightarrow E^1$ and $x_0 \in [a, b]$.
The function f is called fuzzy Caputo fractional differentiable of order $0 < \alpha < 1$ at x_0 if there exists an element ${}^cD^\alpha_a f(x_0) \in E^1$ such that:

$${}^cD^\alpha_a f(x_0) = \frac{1}{\Gamma(\alpha)} \int_a^t \frac{f'(s)}{(t - s)^{1-\alpha}} ds \qquad \text{(II.1)}$$

Then we say f is ${}^c[(1) - \alpha] - differentiable$ if f is (1)-differentiable, and f is ${}^c[(2) - \alpha] - differentiable$ if f is (2)-differentiable.

Remark 3: [5] Since$[f(x)]^r = \left[\underline{f}(x;r), \overline{f}(x;r)\right]$ for each $r \in [0, 1]$,then
1) If f is (1)-differentiable then,
$\left[{}^cD^\alpha_{a+} f(t)\right]^r = \left[{}^cD^\alpha_{a+}\underline{f}(x;r), {}^cD^\alpha_{a+}\overline{f}(x;r)\right]$
2)If f is (2)-differentiable then,
$\left[{}^cD^\alpha_{a+} f(t)\right]^r = \left[{}^cD^\alpha_{a+}\overline{f}(x;r), {}^cD^\alpha_{a+}\overline{f}(x;r)\right]$
Where ${}^cD^\alpha_{a+}\underline{f}(t;r) := \frac{1}{\Gamma(\alpha)} \int_a^t \frac{\underline{f}'(s;r)}{(t-s)^{1-\alpha}} ds$
And
${}^cD^\alpha_{a+}\overline{f}(t;r) := \frac{1}{\Gamma(\alpha)} \int_a^t \frac{\overline{f}'(s;r)}{(t-s)^{1-\alpha}} ds$

III. MAIN RESULTS

We start by writing the boundary value problem (I.1)(I.2) in its integral form.

Lemma 1: Suppose that f is a continuous function. A function $x \in C^F([a;b])$ is a solution of $(I.1)$ $(I.2)$ if and only if x satisfes the integral equation

$$x(t) = \frac{(t-a)^{\alpha-1}}{(b-a)^{\alpha-1}}B + \int_a^b G(t,s)f(t,y(s))ds \qquad \text{(III.1)}$$

$$G(t,s) = \begin{cases} \frac{(t-a)^{\alpha-1}}{(b-a)^{\alpha-1}}(b-s)^{\alpha-1} - (t-s)^{\alpha-1} & a \le s \le t \le b \\ \frac{(t-a)^{\alpha-1}}{(b-a)^{\alpha-1}}(b-s)^{\alpha-1} & a \le t \le s \le b \end{cases}$$

Proof 1:

Proof 2: By using the parametric form of fuzzy number we have $x(t) = (\underline{x}(r), \overline{x}(r))$,then the problem (1.1) ,(1.2) is equivalent to

$$(1.3)\begin{cases} D^\alpha \underline{x}(t;r) = \underline{f}(t, \underline{x}(t;r);r) & t \in [a,b] \\ \underline{x}(a;r) = \underline{0}(r), \quad \underline{x}(b;r) = \underline{B}(r) \end{cases}$$

$$(1.4)\begin{cases} D^\alpha \overline{x}(t;r) = \overline{f}(t, \overline{x}(t;r);r) & t \in [a,b] \\ \overline{x}(a;r) = \overline{0}(r), \quad \overline{x}(b;r) = \overline{B}(r) \end{cases}$$

It is well known that solving $(I.3)$ is equivalent to solving the integral equation
$\underline{x}(t;r) = c\frac{(t-a)^{\alpha-1}}{\Gamma(\alpha)} + d\frac{(t-a)^{\alpha-2}}{\Gamma(\alpha-2)} + \int_a^t G(t,s)\underline{f}(t,\underline{x}(t;r);r)ds$
where c and d are some real constants.Now,$d = 0$ by the first boundary condition.On the other hand, $\underline{x}(b;r) = \underline{B}(r)$ implies.
$\underline{B}(r) = c\frac{(b-a)^{\alpha-1}}{\Gamma(\alpha)} + \int_a^b (b-s)^{\alpha-1}\underline{f}(s,\underline{x}(s;r);r)ds$
which after some manipulations yields
$c = \frac{\Gamma(\alpha)}{(b-a)^{\alpha-1}}\left(\underline{B}(r) - \int_a^b (b-s)^{\alpha-1}\underline{f}(s,\underline{x}(s;r);r)ds\right)$.
$\underline{x}(t;r) = \frac{\Gamma(\alpha)}{(b-a)^{\alpha-1}}\left(\underline{B}(r) - \int_a^b (b-s)^{\alpha-1}\underline{f}(s,\underline{x}(s;r);r)ds\right)$
$\times\frac{(t-a)^{\alpha-1}}{\Gamma(\alpha)} - \int_a^t (t-s)^{\alpha-1}\underline{f}(t,\underline{x}(t;r);r)ds$
and the proof is complete.

Proposition 3: Let G be the Green function given in Lemma (1) Then

$$\int_a^b \mid G(t,s) \mid ds \le \frac{(\alpha-1)^{\alpha-1}}{\Gamma(\alpha)\alpha^{\alpha+1}}(b-a)^{\alpha-1}$$

Proof 3: It is known [2]Lemma (2.2) that $G(t,s) \ge 0$ for all $t,s \in [a,b]$ Therefore
$\int_a^b \mid G(t,s) \mid ds =$
$\frac{1}{\Gamma(\alpha)}\int_a^t \left(\frac{(t-a)^{\alpha-1}}{(b-a)^{\alpha-1}}(b-s)^{\alpha-1} - (t-s)^{\alpha-1}\right)ds$
$+\frac{1}{\Gamma(\alpha)}\int_t^b \left(\frac{(t-a)^{\alpha-1}}{(b-a)^{\alpha-1}}(b-s)^{\alpha-1}\right)ds$
$= \frac{1}{\Gamma(\alpha)}\left(-\frac{(t-a)^{\alpha-1}}{(b-a)^{\alpha-1}}\frac{(b-t)^{\alpha-1}}{\alpha} + \frac{(t-a)^{\alpha-1}}{(b-a)^{\alpha}}\frac{(t-a)^{\alpha-1}}{(b-a)^{\alpha-1}}\frac{(b-a)^{\alpha-1}}{\alpha}\right)$
$+\frac{1}{\Gamma(\alpha)}\left(-\frac{(t-a)^{\alpha-1}}{\alpha} + \frac{(t-a)^{\alpha-1}}{(b-a)^{\alpha-1}}\frac{(b-t)^{\alpha-1}}{\alpha}\right)$
$= \frac{1}{\Gamma(\alpha)}\frac{(t-a)^{\alpha-1}(b-t)}{\alpha}$
We define $g : [a,b] \to R$ by

$$g(t) = \frac{(t-a)^{\alpha-1}(b-t)}{\alpha}$$

Diferentiating the function g we immediately find that its maximum is achieved at the point

$$t^* = \frac{(\alpha-1)b + a}{\alpha}$$

Moreover

$$g(t^*) = \frac{(a-1)^{\alpha-1}(b-a)^\alpha}{\alpha}$$

wich complete the proof

Theorem 2: Assume$f : [a;b] \times E^1 \longrightarrow R$ is continuous and satisfes a uniform Lipschitz condition with respect to the second variable on $[a,b] \times E^1$ with Lipschitz constant K that is,

$$D\left(f(t,x(t)), f(t,y(t)\right) \le KD(x,y)$$

For all $(t,x),(t,y) \in [a;b] \times E^1$.
If $\frac{K(\alpha-1)^{\alpha-1}}{\Gamma(\alpha)\alpha^{\alpha+1}}(b-a)^{\alpha-1} < 1$ then the boundary-value problem

$$D^\alpha x(t) = f(t,x(t)) \quad t \in [a,b]$$

$$x(a) = \tilde{0}, x(b) = B \quad B \in E^1$$

has a unique continuous solution.

Proof 4: Let C^F the complete metric space of fuzzy continuous functions defined on $[a,b]$ with the distance D By Lemma (2.1),$y \in C^F$ is a solution of $(II.2)$,$(II.3)$ if and only if it is a solution of the integral equation

$$y(t) = \frac{(t-a)^{\alpha-1}}{(b-a)^{\alpha-1}}B + \int_a^b G(t,s)f(t,y(s))ds$$

Let $T : C^F \longrightarrow C^F$ defined by

$$Ty(t) = \frac{(t-a)^{\alpha-1}}{(b-a)^{\alpha-1}}B + \int_a^b G(t,s)f(t,y(s))ds$$

for $t \in [a;b]$.We will show that the operator T has a unique fixed point.
Let $x,y \in C^F$,then

$$\begin{aligned} D(Tx(t),Ty(t)) &\le \int_a^b |G(s,t)|D(f(x(s),s),f(y(s),s)))ds \\ &\le \int_a^b |G(s,t)|KD(x(s),y(s))ds \\ &\le \frac{K(\alpha-1)^{\alpha-1}}{\Gamma(\alpha)\alpha^{\alpha+1}}(b-a)^{\alpha-1}D(x,y) \end{aligned}$$

where we have used Proposition 3 By (2.1) we conclude that T is a contracting mapping on C_F and by the Banach contraction mapping theorem we get the desired result.

Remark 4: We note that when $\alpha = 2$ in Theorem 2, one immediately obtains Theorem 1 (apart from the restriction $A = 0$ $(y(a) = 0)$, which we have to assume in order to consider continuous solutions on $[a,b]$ to $(II.2)$.

Example 1: As an example we consider the initial-value problem

$$D^{3/2}(\underline{x}(t;r), \overline{x}(t;r)) = (sin(\underline{x}(t;r)), sin(\overline{x}(t;r))) \quad t \in [0,1] \qquad \text{(III.2)}$$

$$(\underline{x}(0;r), \overline{x}(0;r)) = (0,0) \quad (\underline{x}(1;r), \overline{x}(1;r)) = 0. \qquad \text{(III.3)}$$

Here $f(t,x(t;r);r) = sin(\underline{x}(t;r))$.
And $|sin(x(t))| \leq 1 = K$.
Since $\alpha = 3/2$ we have.

$$\frac{1(\alpha-1)^{\alpha-1}}{\Gamma(\alpha)\alpha^{\alpha+1}}(1-0)^{\alpha-1} = \frac{3}{4}\pi^{\frac{1}{3}}3^{2/3}$$

The condition of theorem 2 is satisfied,thus the initial-value problem $(III.2)$ $(III.3)$ has a unique solution.

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