Numerical solution of intuitionistic fuzzy differential equations by Diagonally implicit block backward differentiation formulas

Bouchra Ben Amma *, Said Melliani * & Lalla Saadia Chadli *
* Laboratory of Applied Mathematics and Scientific Computing, Sultan Moulay Slimane University, B.P523, Beni Mellal, Morocco,
Email: bouchrabenamma@gmail.com, {s.melliani, sa.chadli}@yahoo.fr

Abstract

In this work, the diagonally implicit block backward differentiation formulas (DIBBDF) is developed for solving intuitionistic fuzzy differential equations (IFDEs) under the interpretation of Hukuhara differentiability. The applicability and accuracy of the proposed method has been demonstrated by an example. It is clearly shown that the proposed method obtains good numerical results and suitable for solving IFDEs.

Index Terms

Numerical solution, Block method, Diagonally implicit.

I. Introduction

Fuzzy set theory was proposed by Zadeh in 1965 [21], and it has been applied in various fileds. The principe of the fuzzy set is an extension of the characteristic function taking the value of zero or one to the membership function which can take any value from the interval $[0,1]$. Therefore, the membership function is only a single-valued function, which cannot be used to express the evidences of support and objection simultaneously in many situations. As the fuzzy set cannot be used to make totally explicit all the information or data in such a problem, it confronts a variety of limits in actual applications. For that reason, Atanassov in 1983 and 1986 [1], [2] extended the fuzzy set characterized by a membership function to the intuitionistic fuzzy sets (IFSs), which is characterized by a membership function, a non-membership function, and a degree of uncertainty. Then, the IFSs can describe the fuzzy characters of things in a more detailed and comprehensive way, which is found to be more effective in dealing with vagueness and uncertainty. The intuitionistic fuzzy set has already achieved great success in theoretical research and real applications [18]–[20].

Differential equations with uncertainty plays serve as mathematical models in many fields such as science, physics, economics, psychology, defense and demography. This type of differential equations is called intuitionistic fuzzy differential equations (IFDEs). The topic of intuitionistic fuzzy differential equations (IFDEs) have been rapidly growing in recent years. The first attempt to treat IFDEs has been done in [14]. In recent years, the authors have focused on existence-uniqueness results for intuitionistic fuzzy solutions of some types of IFDEs. They have defined the concept of intuitionistic fuzzy solutions and introduced conditions for existence and uniqueness results using different techniques [6], [7], [9], [10], [12], [16]. The still-standing problem in the theory of intuitionistic fuzzy differential equations is to find implementable numerical methods. Much more effort has been made in this direction as well. There are some applications of numerical methods are introduced in [3]–[5], [8], [11], [13], [17]. In this paper, intuitionistic fuzzy differential equation is solved numerically by diagonally implicit block backward differentiation formulas under the interpretation of Hukuhara differentiability.

The paper is organized as follows: In Section 2, some basic definitions and the IFDE is defined. The derivation and the implementation of implicit block backward differentiation formulas for IFDEs are presented Section 3. The results of these numerical methods are discussed in section 4. The final section is the conclusion.

II. Preliminaries

A. Notations and definitions

Throughout this paper, $(\mathbb{R}, B(\mathbb{R}), \mu)$ denotes a complete finite measure space.
Let us $P_k(\mathbb{R})$ the set of all nonempty compact convex subsets of $\mathbb{R}$.
we denote by

$$\mathbb{F}_1 = \mathrm{IF}(\mathbb{R}) = \{\langle u, v\rangle \,:\, \mathbb{R} \to [0,1]^2, |\forall\, x \in \mathbb{R}\ 0 \le u(x)+v(x) \le 1\}$$

An element $\langle u, v\rangle$ of $\mathbb{F}_1$ is said an intuitionistic fuzzy number if it satisfies the following conditions

(i) $\langle u, v\rangle$ is normal i.e there exists $x_0,\ x_1 \in \mathbb{R}$ such that $u(x_0) = 1$ and $v(x_1) = 1$.
(ii) u is fuzzy convex and v is fuzzy concave.
(iii) u is upper semi-continuous and v is lower semi-continuous
(iv) $supp\langle u, v\rangle = \mathrm{cl}\{x \in \mathbb{R} \ :|\ v(x) < 1\}$ is bounded.

so we denote the collection of all intuitionistic fuzzy number by $\mathbb{F}_1$.

Definition 2.1: Let $\langle u,v\rangle$ an element of $\mathbb{F}_1$ and $\alpha\in[0,1]$, we define the following sets :

$$\Big[\langle u,v\rangle\Big]_l^+(\alpha)=\inf\{x\in\mathbb{R}\mid u(x)\geq\alpha\},$$
$$\Big[\langle u,v\rangle\Big]_r^+(\alpha)=\sup\{x\in\mathbb{R}\mid u(x)\geq\alpha\}$$
$$\Big[\langle u,v\rangle\Big]_l^-(\alpha)=\inf\{x\in\mathbb{R}\mid v(x)\leq 1-\alpha\},$$
$$\Big[\langle u,v\rangle\Big]_r^-(\alpha)=\sup\{x\in\mathbb{R}\mid v(x)\leq 1-\alpha\}$$

Remark 2.1:

$$\Big[\langle u,v\rangle\Big]_\alpha=\Big[\Big[\langle u,v\rangle\Big]_l^+(\alpha),\Big[\langle u,v\rangle\Big]_r^+(\alpha)\Big]$$
$$\Big[\langle u,v\rangle\Big]^\alpha=\Big[\Big[\langle u,v\rangle\Big]_l^-(\alpha),\Big[\langle u,v\rangle\Big]_r^-(\alpha)\Big]$$

A Triangular Intuitionistic Fuzzy Number (TIFN) $\langle u,v\rangle$ is an intuitionistic fuzzy set in $\mathbb{R}$ with the following membership function u and non-membership function v :

$$u(x)=\begin{cases}\frac{x-a_1}{a_2-a_1} & \text{if } a_1\leq x\leq a_2\\ \frac{a_3-x}{a_3-a_2} & \text{if } a_2\leq x\leq a_3\\ 0 & \text{otherwise}\end{cases}$$

$$v(x)=\begin{cases}\frac{a_2-x}{a_2-a_1'} & \text{if } a_1'\leq x\leq a_2\\ \frac{x-a_2}{a_3'-a_2} & \text{if } a_2\leq x\leq a_3'\\ 1 & \text{otherwise}\end{cases}$$

where $a_1'\leq a_1\leq a_2\leq a_3\leq a_3'$ and $u(x),v(x)\leq 0.5$ for $u(x)=v(x),\quad\forall x\in\mathbb{R}$
This TIFN is denoted by $\langle u,v\rangle$=$\langle a_1,a_2,a_3;a_1',a_2,a_3'\rangle$ where,

$$[\langle u,v\rangle]_\alpha=[a_1+\alpha(a_2-a_1),a_3-\alpha(a_3-a_2)] \qquad (1)$$
$$[\langle u,v\rangle]^\alpha=[a_1'+\alpha(a_2-a_1'),a_3'-\alpha(a_3'-a_2)] \qquad (2)$$

Theorem 2.1: ([15])
d_∞ define a metric on IF_1.

Theorem 2.2: The metric space (IF_1,d_∞) is complete.

Remark 2.2: If $F:[a,b]\to\mathbb{F}_1$ is Hukuhara differentiable and its Hukuhara derivative F' is integrable over $[0,1]$ then

$$F(t)=F(t_0)+\int_{t_0}^{t}F'(s)ds$$

Definition 2.2: Let $\langle u,v\rangle$ and $\langle u',v'\rangle\in IF_1$, the H-difference is the IFN $\langle z,w\rangle\in IF_1$, if it exists, such that

$$\langle u,v\rangle-\langle u',v'\rangle=\langle z,w\rangle\Longleftrightarrow\langle u,v\rangle=\langle u',v'\rangle+\langle z,w\rangle$$

Definition 2.3: A mapping $F:[a,b]\to IF_1$ is said to be Hukuhara derivable at t_0 if there exist $F'(t_0)\in IF_1$ such that both limits:

$$\lim_{\Delta t\to 0^+}\frac{F(t_0+\Delta t)-F(t_0)}{\Delta t}$$

and

$$\lim_{\Delta t\to 0^+}\frac{F(t_0)-F(t_0-\Delta t)}{\Delta t}$$

exist and they are equal to $F'(t_0)=\langle u'(t_0),v'(t_0)\rangle$, which is called the Hukuhara derivative of F at t_0.

B. Intuitionistic fuzzy Cauchy problem

In this section, we consider the initial value problem for the intuitionistic fuzzy differential equation

$$\begin{cases}x'(t) &= f(t,x(t)),\quad t\in I\\ x(t_0) &= \langle u_{t_0},v_{t_0}\rangle\in IF_1\end{cases} \qquad (3)$$

where $x\in IF_1$ is unknown $I=[t_0,T]$ and $f:I\times IF_1\to IF_1$ and $x(t_0)$ is intuitionistic fuzzy number.
Denote the $\alpha-$ level set

$$[x(t)]_\alpha=\Big[[x(t)]_l^+(\alpha),[x(t)]_r^+(\alpha)\Big]$$
$$[x(t)]^\alpha=\Big[x(t)]_l^-(\alpha),[x(t)]_r^-(\alpha)\Big]$$
$$[x(t_0)]_\alpha=\Big[[x(t_0)]_l^+(\alpha),[x(t_0)]_r^+(\alpha)\Big]$$
$$[x(t_0)]^\alpha=\Big[x(t_0)]_l^-(\alpha),[x(t_0)]_r^-(\alpha)\Big]$$

and

$$[f(t,x(t))]_\alpha=\Big[f_l^+(t,x(t);\alpha),[f_r^+(t,x(t);\alpha)\Big]$$
$$[f(t,x(t))]^\alpha=\Big[f_l^-(t,x(t);\alpha),[f_r^-(t,x(t);\alpha)\Big]$$

Sufficient conditions for the existence of an unique solution to Eq. (3) are:

1) Continuity of f
2) Lipschitz condition: for any pair $(t,x_1),(t,x_2)\in I\times\mathbb{F}_1$, we have
$$d_\infty\Big(f(t,x_1),f(t,x_2)\Big)\leq Kd_\infty(x_1,x_2) \qquad (4)$$
where $K>0$ is a given constant.

Theorem 2.3: [6] Let us suppose that the following conditions hold:

(a) Let $R_0=[t_0,t_0+p]\times\overline{B}(\langle u,v\rangle_{t_0},q)$, $p,q\geq 0$,$\langle u,v\rangle_{t_0}\in IF_1$ where $\overline{B}(\langle u,v\rangle_{t_0},q)=\{\langle u,v\rangle\in IF_1:d_\infty(\langle u,v\rangle,\langle u,v\rangle_{t_0})\leq q\}$ denote a closed ball in IF_1 and let $f:R_0\longrightarrow IF_1$ be a continuous function such that $d_\infty\big(f(t,\langle u,v\rangle),0_{(1,0)}\big)\leq M$ for all $(t,\langle u,v\rangle)\in R_0$.
(b) Let $g:[t_0,t_0+p]\times[0,q]\longrightarrow\mathbb{R}$ such that $g(t,0)\equiv 0$ and $0\leq g(t,x)\leq M_1$, $\forall t\in[t_0,t_0+p],0\leq x\leq q$ such that $g(t,x)$ is non-decreasing in u and g is such that the initial value problem
$$x'(t)=g(t,x(t)),x(t_0)=x_0. \qquad (5)$$
has only the solution $x(t)\equiv 0$ on $[t_0,t_0+p]$
(c) We have $d_\infty\Big(f\big(t,\langle u,v\rangle\big),f\big(t,\langle z,w\rangle\big)\Big)\leq g(t,d_\infty(\langle u,v\rangle,\langle z,w\rangle))$, $\forall(t,\langle u,v\rangle)$, $(t,\langle z,w\rangle)\in R_0$ and $d_\infty(\langle u,v\rangle,\langle z,w\rangle)\leq q$. Then the intuitionistic fuzzy initial value problem
$$\begin{cases}\langle u,v\rangle' &= f(t,\langle u,v\rangle),\\ \langle u,v\rangle(t_0) &= \langle u,v\rangle_{t_0}\end{cases} \qquad (6)$$

has an unique solution $\langle u,v\rangle \in C^1[[t_0,t_0+r],B(x_0,q)]$ on $[t_0,t_0+r]$ where $r=\min\{p,\frac{q}{M},\frac{q}{M_1},d\}$ and the successive iterations

$$\langle u,v\rangle_0(t)=\langle u,v\rangle_{t_0},$$
$$\langle u,v\rangle_{n+1}(t)=\langle u,v\rangle_{t_0}+\int_{t_0}^{t} f(s,\langle u,v\rangle_n(s))ds$$

converge to $\langle u,v\rangle(t)$ on $[t_0,t_0+r]$.

C. Interpolation of intuitionistic fuzzy number

The problem of interpolation for intuitionistic fuzzy sets is as follows:
Suppose that at various time instant x information $f(x)$ is presented as intuitionistic fuzzy set. The aim is to approximate the function $f(x)$, for all x in the domain of f. Let $x_0<x_1<\ldots<x_n$ be $n+1$ distinct points in $\mathbb{R}$ and let $\langle u_0,v_0\rangle,\langle u_1,v_1\rangle,\ldots,\langle u_n,v_n\rangle$ be $n+1$ intuitionistic fuzzy sets in IF_1. An intuitionistic fuzzy polynomial interpolation of the data is an intuitionistic fuzzy-value continuous function $f:I\longrightarrow IF_1$ satisfying:

- $f(x_i)=\langle u_i,v_i\rangle$
- If the data is crisp, then the interpolation f is a crisp polynomial.

A function f which fulfilling these condition may be constructed as follows. For each $Y=(y_0,y_1,\ldots,y_n)\in\mathbb{R}^{n+1}$, the unique polynomial of degree $\leq n$ denoted by P_Y such that

- $P_Y(x_i)=y_i,\quad i=0,1,\ldots,n$
- $P_Y(x)=\sum_{i=0}^{n} y_i\Big(\prod\limits_{i\neq j}\frac{x-x_j}{x_i-x_j}\Big)$

According to the extension principle, we can write the membership and non-membership function $f(x)$ for each $x\in\mathbb{R}$ as follows:

$$\mu_{f(x)}(t)=\begin{cases}\sup\limits_{\substack{y_0,y_1,\ldots,y_n\\ t=P_{y_0,y_1,\ldots,y_n}(x)}}\ \min\limits_{i=0,1,\ldots,n}\mu_{u_i}(y_i) & \text{if } P^{-1}_{y_0,y_1,\ldots,y_n}(t)\neq\emptyset\\ 0 & \text{otherwise}\end{cases}$$

where μ_{u_i} is the membership function of u_i, and

$$\nu_{f(x)}(t)=\begin{cases}\inf\limits_{\substack{y_0,y_1,\ldots,y_n\\ t=P_{y_0,y_1,\ldots,y_n}(x)}}\ \max\limits_{i=0,1,\ldots,n}\nu_{v_i}(y_i) & \text{if } P^{-1}_{y_0,y_1,\ldots,y_n}(t)\neq\emptyset\\ 1 & \text{otherwise}\end{cases}$$

where ν_{v_i} is the non-membership function of v_i.
Let $J_i^+(\alpha)=[\langle u_i,v_i\rangle]_\alpha$, $J_i^-(\alpha)=[\langle u_i,v_i\rangle]^\alpha$ for any $\alpha\in[0,1]$, $i=0,1,...,n$ and $[f(x)]_\alpha$, $[f(x)]^\alpha$ the upper and lower α-cuts of $\langle u_i,v_i\rangle$ and $f(x)$ respectively. Hence,

$$\begin{aligned}[f(x)]_\alpha &= \big\{t\in\mathbb{R}\ \mid\ \mu_{f(x)}(t)\geq\alpha\big\}\\ &= \big\{t\in\mathbb{R}\mid \exists\, y_0,y_1,\ldots,y_n:\mu_{u_i}(y_i)\geq\alpha,\\ &\qquad i=0,\ldots,n \text{ and } P_{y_0,y_1,\ldots,y_n}(x)=t\big\}\\ &= \big\{t\in\mathbb{R}\mid \exists\, Y\in\prod_{i=0}^{n}J_i^+(\alpha)\\ &\qquad :P_{y_0,y_1,\ldots,y_n}(x)=t\big\}\end{aligned}$$

and

$$\begin{aligned}[f(x)]^\alpha &= \big\{t\in\mathbb{R}\ \mid\ \nu_{f(x)}(t)\leq 1-\alpha\big\}\\ &= \big\{t\in\mathbb{R}\mid \exists\, y_0,y_1,\ldots,y_n:\nu_{v_i}(y_i)\leq 1-\alpha,\\ &\qquad i=0,\ldots,n \text{ and } P_{y_0,y_1,\ldots,y_n}(x)=t\big\}\\ &= \big\{t\in\mathbb{R}\mid \exists\, Y\in\prod_{i=0}^{n}J_i^-(\alpha)\\ &\qquad :P_{y_0,y_1,\ldots,y_n}(x)=t\big\}\end{aligned}$$

Finally, for each $x\in\mathbb{R}$ and all $t\in\mathbb{R}$ is defined by $f(x)\in IF_1$ by

$$f(x)(t)=\Big(\sup\big\{\alpha\in(0,1]|\exists Y\in\prod_{i=0}^{n}J_i^+(\alpha):P_Y(x)=t\big\},\ 1-\sup\big\{\alpha\in(0,1]|\exists Y\in\prod_{i=0}^{n}J_i^-(\alpha):P_Y(x)=t\big\}\Big)$$

where $Y=(y_0,y_1,\ldots,y_n)\in\mathbb{R}^{n+1}$
The interpolation polynomial can be written level set wise as

$$\begin{aligned}[f(x)]_\alpha &= \{y\in\mathbb{R}:y=P_{y_0,y_1,\ldots,y_n}(x),\\ &\qquad y_i\in[\langle u_i,v_i\rangle]_\alpha\quad i=0,\ldots,n\},\quad \text{for}\quad \alpha\in(0,1]\end{aligned}$$

and

$$\begin{aligned}[f(x)]^\alpha &= \{y\in\mathbb{R}:y=P_{y_0,y_1,\ldots,y_n}(x),\\ &\qquad y_i\in[\langle u_i,v_i\rangle]^\alpha\quad i=0,\ldots,n\},\quad \text{for}\quad \alpha\in(0,1]\end{aligned}$$

But, from Lagrange interpolation formula, we have

$$[f(x)]_\alpha=\sum_{i=0}^{n}\ell_i(x)J_i^+(\alpha)$$

and

$$[f(x)]^\alpha=\sum_{i=0}^{n}\ell_i(x)J_i^-(\alpha)$$

where $\ell_i(x)$ represents the Lagrange polynomials.
When the data $\langle u_i,v_i\rangle$ presents as triangular intuitionistic fuzzy numbers, values of the interpolation polynomial are also triangular intuitionistic fuzzy numbers. Then $f(x)$ has a particular simple form that is well suited to computation. Denote $J_i^+(\alpha)=[a_i^+(\alpha),b_i^+(\alpha)]$ and $J_i^-(\alpha)=[a_i^-(\alpha);b_i^-(\alpha)]$. Then the upper end point of $[f(x)]_\alpha$ is the solution of the optimization problem :

$$\text{Maximize}\quad P_{y_0,y_1,\ldots,y_n}(x)\quad \text{subject to}\quad a_i^+(\alpha)\leq y_i\leq b_i^+(\alpha),\quad i=0,1,\ldots,n$$

It follows that the optimal solution is

$$y_i = \begin{cases} b_i^+(\alpha) & \text{if} \quad \ell_i(x) \geq 0 \\ a_i^+(\alpha) & \text{if} \quad \ell_i(x) < 0 \end{cases}$$

and the lower end point is obtained as the value of the interpolation polynomial associated to points

$$y_i = \begin{cases} b_i^+(\alpha) & \text{if} \quad \ell_i(x) < 0 \\ a_i^+(\alpha) & \text{if} \quad \ell_i(x) \geq 0 \end{cases}$$

Similarly the upper and lower end point of $[f(x)]^\alpha$ can be obtained.
Hence if $\langle u_i, v_i \rangle$ is an intuitionistic fuzzy number, for all i then also $f(x)$ is such an intuitionistic fuzzy number for each x. More precisely, if $\langle u_i, v_i \rangle = \langle u_i^l, u_i^c, u_i^r, v_i^l, u_i^c, v_i^r \rangle$ and $f(x) = \big\langle f_l(x), f^c(x), f_r(x), f^l(x), f^c(x), f^r(x) \big\rangle$, then we will have,

$$\begin{aligned} f_l(x) &= \sum_{\ell_i(x)\geq 0} \ell_i(x)u_i^l + \sum_{\ell_i(x)<0} \ell_i(x)u_i^r \\ f_r(x) &= \sum_{\ell_i(x)\geq 0} \ell_i(x)u_i^r + \sum_{\ell_i(x)<0} \ell_i(x)u_i^l \\ f^c(x) &= \sum_{i=0}^{n} \ell_i(t)u_i^c \\ f^l(x) &= \sum_{\ell_i(x)\geq 0} \ell_i(x)v_i^l + \sum_{\ell_i(x)<0} \ell_i(x)v_i^r \\ f^r(x) &= \sum_{\ell_i(x)\geq 0} \ell_i(x)v_i^r + \sum_{\ell_i(x)<0} \ell_i(x)v_i^l \end{aligned}$$

III. Formulation of Diagonally Implicit Block Backward Differentiation Formulas

The fully implicit block backward differentiation was derived using Lagrange polynomial, $P_k(t)$ of degree k which interpolates the values $x_n, x_{n-1}, \ldots, x_{n-k+1}$ of a function f at interpolating points $t_n, t_{n-1}, t_{n+1}, \ldots, t_{n-k+1}$ in terms of Lagrange polynomial defined as follows:

$$P(t) = \sum_{j=0}^{k} \ell_{k,j}(t) f(t_{n+1-j})$$

where $\ell_{k,j}(t) = \prod_{\substack{i=0 \\ i\neq j}}^{k} \frac{(t - t_{n+1-i})}{(t_{n+1-j} - t_{n+1-i})}$, for $j = 0, 1, \ldots, k$.
Next, we defined $s = \frac{t - t_{n+1}}{h}$ and replace $f(t, x)$ by polynomial (6) which interpolates only the values x_{n-1}, x_n and x_{n+1} at the interpolating points $t_{n-1}, t_n, \ldots, t_{n+1}$ to compute x_{n+1}.

$$\begin{aligned} P(t) = \quad & \tfrac{(t_{n+1}+sh-t_n)(t_{n+1}+sh-t_{n+1})}{(-h)(-2h)} x(t_{n-1}) \\ & + \tfrac{(t_{n+1}+sh-t_{n-1})(t_{n+1}+sh-t_{n+1})}{(h)(-h)} x(t_n) \\ & + \tfrac{(t_{n+1}+sh-t_{n-1})(t_{n+1}+sh-t_n)}{(2h)(h)} x(t_{n+1}) \end{aligned}$$

Thus, differentiating the resulting polynomial once with respect to s at the point $t = t_{n+1}$ and evaluating at $s = 0$ gives the following corrector formula for first point as follows

$$x_{n+1} = -\frac{1}{3}x_{n-1} + \frac{4}{3}x_n + \frac{2}{3}hf(t_{n+1}, x_{n+1}) \qquad (7)$$

Then, we interpolate the values x_{n-1}, x_n, x_{n+1} and x_{n+2} and at the interpolating points $t_{n-1}, t_n, t_{n+1}, \ldots, t_{n+2}$ to compute x_{n+2} and differentiating the resulting polynomial once with respect to s at the point $t = t_{n+2}$ and substituting $s = 0$ yields

$$x_{n+1} = \frac{2}{11}x_{n-1} - \frac{9}{11}x_n + \frac{18}{11}x_{n+1} + \frac{6}{11}hf(t_{n+2}, x_{n+2}) \quad (8)$$

Let the exact solutions

$$[X(t_n)]_\alpha = \Big[[X(t_n)]_l^+(\alpha), [X(t_n)]_r^+(\alpha)\Big]$$

$$[X(t_n)]^\alpha = \Big[[X(t_n)]_l^-(\alpha), [X(t_n)]_r^-(\alpha)\Big]$$

be approximated by

$$\begin{aligned} [x(t_n)]_\alpha &= \Big[[x(t_n)]_l^+(\alpha), [x(t_n)]_r^+(\alpha)\Big] \\ [x(t_n)]^\alpha &= \Big[[x(t_n)]_l^-(\alpha), [x(t_n)]_r^-(\alpha)\Big] \end{aligned} \qquad (9)$$

at t_n, $0 \leq n \leq N$
The solutions are calculated by grid points at

$$t_0 < t_1 < t_2 < \ldots < t_N = T, \quad h = \frac{T - t_0}{N}, \quad t_n = t_0 + nh, \quad n = 0, 1, \ldots N$$

By using formula (7) and (8), we configure the general intuitionistic fuzzy DIBBDF for approximates solutions as follows:

$$\begin{cases} [x(t_{n+1})]_l^+(\alpha) = -\frac{1}{3}[x(t_{n-1})]_l^+(\alpha) + \frac{4}{3}[x(t_n)]_l^+(\alpha) \\ \qquad + \frac{2}{3}hf_l^+(t_{n+1}, x(t_{n+1}); \alpha) \\ [x(t_{n+1})]_r^+(\alpha) = -\frac{1}{3}[x(t_{n-1})]_r^+(\alpha) + \frac{4}{3}[x(t_n)]_r^+(\alpha) \\ \qquad + \frac{2}{3}hf_r^+(t_{n+1}, x(t_{n+1}); \alpha) \\ [x(t_{n+1})]_l^-(\alpha) = -\frac{1}{3}[x(t_{n-1})]_l^-(\alpha) + \frac{4}{3}[x(t_n)]_l^-(\alpha) \\ \qquad + \frac{2}{3}hf_l^-(t_{n+1}, x(t_{n+1}); \alpha) \\ [x(t_{n+1})]_r^-(\alpha) = -\frac{1}{3}[x(t_{n-1})]_r^-(\alpha) + \frac{4}{3}[x(t_n)]_r^-(\alpha) \\ \qquad + \frac{2}{3}hf_r^-(t_{n+1}, x(t_{n+1}); \alpha) \end{cases}$$

and

$$\begin{cases} [x(t_{n+2})]_l^+(\alpha) = \frac{2}{11}[x(t_{n-1})]_l^+(\alpha) - \frac{9}{11}[x(t_n)]_l^+(\alpha) \\ \qquad + \frac{18}{11}[x(t_{n+1})]_l^+(\alpha) + \frac{6}{11}hf_l^+(t_{n+2}, x(t_{n+2}); \alpha) \\ [x(t_{n+2})]_r^+(\alpha) = \frac{2}{11}[x(t_{n-1})]_r^+(\alpha) - \frac{9}{11}[x(t_n)]_r^+(\alpha) \\ \qquad + \frac{18}{11}[x(t_{n+1})]_r^+(\alpha) + \frac{6}{11}hf_r^+(t_{n+2}, x(t_{n+2}); \alpha) \\ [x(t_{n+2})]_l^-(\alpha) = \frac{2}{11}[x(t_{n-1})]_l^-(\alpha) - \frac{9}{11}[x(t_n)]_l^-(\alpha) \\ \qquad + \frac{18}{11}[x(t_{n+1})]_l^-(\alpha) + \frac{6}{11}hf_l^-(t_{n+2}, x(t_{n+2}); \alpha) \\ [x(t_{n+2})]_r^-(\alpha) = \frac{2}{11}[x(t_{n-1})]_r^-(\alpha) - \frac{9}{11}[x(t_n)]_r^-(\alpha) \\ \qquad + \frac{18}{11}[x(t_{n+1})]_r^-(\alpha) + \frac{6}{11}hf_r^-(t_{n+2}, x(t_{n+2}); \alpha) \end{cases}$$

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IV. Example

In this section, one set of intuitionistic fuzzy initial value problem is tested for the purpose of validating the difference in numerical results. The test problem and solution are listed below.

Example 4.1: Consider the intuitionistic fuzzy initial value problem

$$\begin{cases} x'(t) &= x(t) \quad \text{for all} \quad t \in [0, T] \\ x_0 &= \Big((\alpha - 1, 1 - \alpha), (-2\alpha, 2\alpha)\Big) \end{cases} \tag{10}$$

Then, we have the following parametrized differential system:

$$\begin{cases} [x(t)]_l^+(\alpha) = & (\alpha - 1)\exp(t) \\ [x(t)]_r^+(\alpha) = & (1 - \alpha)\exp(t) \\ [x(t)]_l^-(\alpha) = & -2\alpha\exp(t) \\ [x(t)]_r^-(\alpha) = & 2\alpha\exp(t) \end{cases}$$

Therefore the exact solutions are given by

$$[X(t)]_\alpha = \Big[(\alpha - 1)\exp(t), (1 - \alpha)\exp(t)\Big]$$

$$[X(t)]^\alpha = \Big[-2\alpha\exp(t), 2\alpha\exp(t)\Big]$$

which at $t = 1$ are

$$[X(1)]_\alpha = \Big[(\alpha - 1)\exp(1), (1 - \alpha)\exp(1)\Big]$$

$$[X(1)]^\alpha = \Big[(-2\alpha\exp(1), 2\alpha\exp(1)\Big]$$

Comparison of results of the the DIBBDF method and Runge-Kutta Nyström method in [11] for $h = 0.2$ and $t = 1$:

	Exact	
α	$([X]_l^+, [X]_r^+)$	$([X]_l^-, [X]_r^-)$
0	(-2.718281828,2.718281828)	(0,0)
0.2	(-2.174625462,2.174625462)	(-1.087312731,1.087312731)
0.4	(-1.630969097,1.630969097)	(-2.174625462,2.174625462)
0.6	(-1.087312731,1.087312731)	(-3.261938194,3.261938194)
0.8	(-0.543656365,0.543656365)	(-4.349250925,4.349250925)
1	(0,0)	(-5.436563656,5.436563656)

	DIBBDF		RK-Nyström	
α	$([x]_l^+, [x]_r^+)$	$([x]_l^-, [x]_r^-)$	$([x]_l^+, [x]_r^+)$	$([x]_l^-, [x]_r^-)$
0	(-2.718251136,2.718251136)	(0,0)	(-2.717509377,2.717509377)	(0,0)
0.2	(-2.174600909,2.174600909)	(-1.087300454,1.087300454)	(-2.174007501,2.174007501)	(-1.087003750,1.087003750)
0.4	(-1.630950681,1.630950681)	(-2.174600909,2.174600909)	(-1.630505626,1.630505626)	(-2.174007501,2.174007501)
0.6	(-1.087300454,1.087300454)	(-3.261901363,3.261901363)	(-1.087003750,1.087003750)	(-3.261011252,3.261011252)
0.8	(-0.543650227,0.543650227)	(-4.349201818,4.349201818)	(-0.543501875,0.543501875)	(-4.348015003,4.348015003)
1	(0,0)	(-5.436502273,5.436502273)	(0,0)	(-5.435018754,5.435018754)

α	Error in RK-Nyström	Error in DIBBDF
0	3.8622×10^{-4}	1.5345×10^{-5}
0.2	4.6347×10^{-4}	1.8415×10^{-5}
0.4	5.4071×10^{-4}	2.1484×10^{-5}
0.6	6.1796×10^{-4}	2.4553×10^{-5}
0.8	6.9520×10^{-4}	2.7622×10^{-5}
1	7.7245×10^{-4}	3.0691×10^{-5}

V. Conclusion

In this paper, we have applied a new block method for numerical solution of intuitionistic fuzzy differential equations. The proposed method obtains better accuracy compared to the existing method in terms of error. We can conclude that DIBBDF is one of the efficient methods for solving intuitionistic fuzzy differential equations.

References

[1] K. T. Atanassov, Intuitionistic fuzzy sets. VII ITKR's session, Sofia (deposited in Central Science and Technical Library of the Bulgarian Academy of Sciences 1697/84), 1983.
[2] K. T. Atanassov, Intuitionistic fuzzy sets, Fuzzy Sets and Systems, Vol.20, pp.87-96, 1986.
[3] B. Ben Amma, S. Melliani and L. S. Chadli, Numerical solution of intuitionistic fuzzy differential equations by Euler and Taylor methods, Notes on Intuitionistic Fuzzy Sets, Vol.22, No.2, pp.71-86, 2016.
[4] B. Ben Amma, S. Melliani and L. S. Chadli, Numerical solution of intuitionistic fuzzy differential equations by Adams three order

predictor-corrector method, Notes on Intuitionistic Fuzzy Sets, Vol.22, No.3, pp.47-69, 2016.

[5] B. Ben Amma and L. S. Chadli, Numerical solution of intuitionistic fuzzy differential equations by Runge-Kutta Method of order four, Notes on Intuitionistic Fuzzy Sets, Vol.22, No.4, pp.42-52, 2016.

[6] B. Ben Amma, S. Melliani and L. S. Chadli, The Cauchy Problem of Intuitionistic Fuzzy Differential Equations, Notes on Intuitionistic Fuzzy Sets, Vol.24, No.1, pp.37-47, 2018.

[7] B. Ben Amma, S. Melliani and L. S. Chadli, Intuitionistic Fuzzy Functional Differential Equations, Fuzzy Logic in Intelligent System Design: Theory and Applications, Ed. Cham: Springer International Publishing, pp.335-357, 2018.

[8] B. Ben Amma, S. Melliani and L. S. Chadli, A Fourth Order Runge-Kutta Gill Method for the Numerical Solution of Intuitionistic Fuzzy Differential Equations, Recent Advances in Intuitionistic Fuzzy Logic Systems, Studies in Fuzziness and Soft Computing, Vol.372. Springer, Cham, 2019.

[9] B. Ben Amma, S. Melliani and L. S. Chadli, The Existence and Uniqueness of Intuitionistic Fuzzy Solutions for Intuitionistic Fuzzy Partial Functional Differential Equations, International Journal of Differential Equations, Vol.2019, No.2019, pp.1-13, 2019.

[10] B. Ben Amma, S. Melliani and L. S. Chadli, Integral Boundary Value Problem for Intuitionistic Fuzzy Partial Hyperbolic Differential Equations, Nonlinear Analysis and Boundary Value Problems, Springer Proceedings in Mathematics & Statistics 292, 2019.

[11] B. Ben Amma, S. Melliani and L. S. Chadli, The Numerical Solution Of Intuitionistic Fuzzy Differential Equations By The Third Order Runge-Kutta Nyström Method, Intuitionistic and Type-2 Fuzzy Logic Enhancements in Neural and Optimization Algorithms: Theory and Applications, Springer, Vol.862, 2020.

[12] B. Ben Amma, S. Melliani and L. S. Chadli, Intuitionistic Fuzzy Partial Functional Differential Equations with Integral Boundary Conditions, 2020, IEEE 6th International Conference on Optimization and Applications (ICOA), Beni Mellal, Morocco, pp. 1-5, 2020.

[13] B. Ben Amma, S. Melliani and L. S. Chadli, Numerical Solution of Intuitionistic Fuzzy Differential Equations By Runge-Kutta Verner Method, Recent Advances in Intuitionistic Fuzzy Logic Systems and Mathematics, Studies in Fuzziness and Soft Computing, Springer International Publishing, Vol.395, pp.53-69, 2021.

[14] S. Melliani and L. S. Chadli, Intuitionistic fuzzy differential equation, Notes on Intuitionistic Fuzzy Sets, Vol.6, pp.37-41, 2000.

[15] S. Melliani, M. Elomari, L. S. Chadli and R. Ettoussi, Intuitionistic Fuzzy Metric Space, Notes on Intuitionistic Fuzzy sets, Vol.21, No.1, pp.43-53 (2015).

[16] S. Melliani, M. Elomari, M. Atraoui and L. S. Chadli, Intuitionistic fuzzy differential equation with nonlocal condition, Notes on Intuitionistic Fuzzy sets, Vol.21, No.4, pp.58-68, 2015.

[17] S. Melliani, H. Atti, B. Ben Amma and L. S. Chadli, Solution of n-th order intuitionistic fuzzy differential equation by variational iteration method, Notes on Intuitionistic Fuzzy sets, Vol.24, No.3, pp.92-105, 2018.

[18] M. H. Shu, C. H. Cheng and J. R. Chang, Using intuitionistic fuzzy sets for fault-tree analysis on printed circuit board assembly, Microelectron. Reliab., Vol.46, No.12, pp.2139-2148, 2006.

[19] O. Castillo, Framework for Optimization of Intuitionistic and Type-2 Fuzzy Systems in Control Applications, Recent Advances in Intuitionistic Fuzzy Logic Systems, Studies in Fuzziness and Soft Computing, Springer, Vol.372, 2019.

[20] E. Szmidt, J. Kacprzyk, Intuitionistic fuzzy sets in some medical applications, Computational Intelligence: Theory and Applications, Springer, pp.148-151, 2001.

[21] L. A. Zadeh, Fuzzy sets, Inf. Control, Vol.8, No.3, pp.338-353, 1965.